ECONOMIC PREHISTORY:

SIX TRANSITIONS THAT SHAPED THE WORLD

PROOFS OF FORMAL PROPOSITIONS

Gregory K. Dow and Clyde G. Reed

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This document contains mathematical proofs of all Propositions, Lemmas, and Corollaries from the book, along with proofs of selected Remarks, Comments, and Results.

Please refer to the text of the book for statements of each formal assertion, relevant assumptions and definitions, numbered equations mentioned in the proofs, and numbered graphs or tables.

The proofs are paginated separately for each of the relevant chapters (3, 4, 5, 6, 7, 8, and 10).

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Economic Prehistory: Six Transitions That Shaped The World

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Chapter 3: The Upper Paleolithic

Proofs of Formal Propositions

Proof of Proposition 3.1 (learning by doing).

Choose any resource r. Let $Z \equiv X(n_r^t/N^t)N^{t+1} \equiv Xn_r^t\rho^t$ be the number of copies of string k_r^t, where X is the number of observations per child, n_r^t/N^t is the fraction of these observations that pertain to resource r, and N^{t+1} is the number of children who survive to become adults in period t+1. Let z = 1 . . Z index the individual copies k_{rz} of k_r^t.

The number of digits in k_r^t that match k_r^* is $q_r^t \in \{0, 1 . . Q\}$ and the number of digits in k_{rz} that match k_r^* is $q_{rz} \in \{0, 1 . . Q\}$. The random variables q_{rz} are iid conditional on k_r^t. Fix the string k_r^t and define the probabilities

$$\pi_q(p) \equiv \Pr(q_{rz} = q \mid k_r^t; p) \qquad \text{for all } q = 0, 1 . . Q \text{ and } z = 1 . . Z.$$

Then define

$$\theta_q(p) \equiv \Pr(q_r^{t+1} = q \mid k_r^t; p) = \Pr(\max\{q_{r1} . . q_{rZ}\} = q \mid k_r^t; p)$$

$$= \Pr(q_{rz} \leq q \text{ for } z = 1 . . Z \mid k_r^t; p) - \Pr(q_{rz} \leq q\text{-}1 \text{ for } z = 1 . . Z \mid k_r^t; p)$$

$$= [\pi_0(p) + . . + \pi_q(p)]^Z - [\pi_0(p) + . . + \pi_{q-1}(p)]^Z$$

where $\pi_0(p) + . . + \pi_{q-1}(p) \equiv 0$ for q = 0.

We want to compute each $\theta_q(p)$ when $X \to \infty$ and $p \to 0$ such that $Xp \equiv \lambda > 0$ is constant. From the definition of Z we have $Z(p) = \lambda\rho^t n_r^t/p$. In what follows we drop the r subscript and the t superscripts in this expression. Next define

$$\theta_q^* \equiv \lim_{p\to 0} \theta_q(p)$$

$$= \lim_{p\to 0} [\pi_0(p) + . . + \pi_q(p)]^{Z(p)} - \lim_{p\to 0} [\pi_0(p) + . . + \pi_{q-1}(p)]^{Z(p)}$$

$$= \{\lim_{p\to 0} \exp[(1/p)\ln(\pi_0(p) + . . + \pi_q(p))]\}^{\lambda n\rho}$$

$$- \{\lim_{p\to 0} \exp[(1/p)\ln(\pi_0(p) + . . + \pi_{q-1}(p))]\}^{\lambda n\rho}$$

We have $\lim_{p\to 0} \pi_q(p) = 1$ for $q = q_r^t$ and $\lim_{p\to 0} \pi_q(p) = 0$ for $q \neq q_r^t$ because at least one mutation must occur whenever the number of correct digits differs from q_r^t. This implies that for each expression of the form $\lim_{p\to 0} \exp[(1/p)\ln(\pi_0(p) + .. + \pi_q(p))]$, there are two possible cases:

(a) If $q < q_r^t$ then $\lim_{p\to 0} \exp[(1/p)\ln(\pi_0(p) + .. + \pi_q(p))] = e^{-\infty} = 0$ and hence $\theta_q^* = 0$.

(b) If $q \geq q_r^t$ then $\lim_{p\to 0} (\pi_0(p) + .. + \pi_q(p)) = 1$ and hence $\lim_{p\to 0} \exp[(1/p)\ln(\pi_0(p) + .. \pi_q(p))] = \lim_{p\to 0} \exp[\pi_0'(p) + .. + \pi_q'(p)]$.

Now consider the derivatives $\pi_q'(p)$ in case (b) above. The probabilities $\pi_q(p)$ are polynomials in p, and all outcomes involving two or more mutations correspond to terms that are quadratic or higher. After taking derivatives, all such terms vanish in the limit. Thus, we can confine attention to outcomes that involve either no mutations or just one mutation. This implies that only $q = q_r^t - 1$, $q = q_r^t$, and $q = q_r^t + 1$ are relevant.

(i) $q = q_r^t - 1$. This outcome can be obtained in q_r^t ways by having one mutation at a locus that is correct in period t and no mutations elsewhere, which has probability $q_r^t p(1-p)^{Q-1}$. All other ways to obtain this result involve three or more mutations. This gives $\lim_{p\to 0} \pi_q'(p) = q_r^t$ so $\lim_{p\to 0} \exp[\pi_0'(p) + .. + \pi_q'(p)] = \exp(q_r^t)$.

(ii) $q = q_r^t$. This outcome can be obtained in one way with no mutations, which has probability $(1-p)^Q$. All other ways to obtain the same result involve two or more mutations. This gives $\lim_{p\to 0} \pi_q'(p) = -Q$ so $\lim_{p\to 0} \exp[\pi_0'(p) + .. + \pi_q'(p)] = \exp[-(Q-q_r^t)]$.

(iii) $q = q_r^t + 1$. This outcome can be obtained in $Q-q_r^t$ ways by having one mutation at an incorrect locus and no mutations elsewhere with probability $(Q-q_r^t)p(1-p)^{Q-1}$.

All other ways to obtain the same result involve three or more mutations. This gives $\lim_{p\to 0} \pi_q'(p) = Q\text{-}q_r^t$ and thus $\lim_{p\to 0} \exp[\pi_0'(p) + . . + \pi_q'(p)] = e^0 = 1$. Recall from (a) above that $\theta_q^* = 0$ when $q < q_r^t$. To solve for θ_q^* when $q = q_r^t$ we first observe that the second limit in the last line for θ_q^* is zero due to (a). Substituting from (b) and (ii) in the first limit in the last line for θ_q^* gives $\theta_q^* = \exp[-\lambda n_r^t \rho^t(Q\text{-}q_r^t)]$.

To solve for θ_q^* when $q = q_r^t + 1$ we observe that (b) applies to both limits in the last line for θ_q^*. Substituting from (b) gives $\theta_q^* = 1\text{-} \exp[-\lambda n_r^t \rho^t(Q\text{-}q_r^t)]$.

Finally, we have $\theta_q^* = 0$ when $q \geq q_r^t + 2$ because (b) applies to both limits in the last line for θ_q^* and both of these limits equal unity. The latter result follows from (iii) and the fact that any outcome with $q \geq q_r^t + 2$ requires two or more mutations.

By construction all of the θ_q^* are conditional on k_r^t. However, the structure of the proof shows that the only relevant property of k_r^t is the number of correct digits q_r^t. Thus we can write the transition probabilities for q_r^t and q_r^{t+1} as in Proposition 3.1.

The limiting transition probabilities for strings, $\lim_{p\to 0} \Pr(k_r^{t+1} = k \mid k_r^t; p)$, follow from the preceding results. If k has fewer correct digits than q_r^t or more than $q_r^t + 1$, it has probability zero in the limit. The only way to have $q_r^{t+1} = q_r^t$ in the limit is by having $k_r^{t+1} = k_r^t$ so this has probability $\exp[-\lambda n_r^t \rho^t(Q\text{-}q^t)]$. There are $Q\text{-}q_r^t$ ways to obtain $q_r^{t+1} = q_r^t + 1$ by a single mutation that changes one incorrect digit to a correct one. Each of these strings k_r^{t+1} has probability $\{1\text{-} \exp[-\lambda n_r^t \rho^t(Q\text{-}q_r^t)]\}/(Q\text{-}q_r^t)$. This ends the proof.

Proof of Proposition 3.2 (short-run equilibrium).

Uniqueness follows from strict concavity of the objective function and continuity follows from the theorem of the maximum. Note that due to the strict concavity, the first order conditions are sufficient for a solution.

(a) Let n' be optimal for (A, N') and let n'' be optimal for (A, N''). Suppose that $n_r'' \leq n_r'$. The first order conditions for (3.2) with parameters (A, N'') imply that all s with $n_s'' > 0$ have $A_s f_s'(n_s'') \geq A_r f_r'(n_r'')$. The fact that $n_r'' \leq n_r'$ gives $A_r f_r'(n_r'') \geq A_r f_r'(n_r')$. Finally, the first order conditions for (3.2) with parameters (A, N') and $n_r' > 0$ give $A_r f_r'(n_r') \geq A_s f_s'(n_s')$ for all s = 1 . . R. This series of inequalities gives $A_s f_s'(n_s'') \geq A_s f_s'(n_s')$ for all s with $n_s'' > 0$ and thus implies $n_s' \geq n_s''$ for all s such that $n_s'' > 0$. Clearly $n_s' \geq n_s''$ also holds for all s such that $n_s'' = 0$. Summing over resources gives $N' \geq N''$, contradicting the assumption $N' < N''$. This shows that $n_r'' > n_r'$.

(b) Let n' be optimal for (A', N) and let n'' be optimal for (A'', N). Suppose $n_r'' \leq n_r'$. For all $v \neq r$ such that $n_v'' > 0$ we have $A_v'' f_v'(n_v'') \geq A_r'' f_r'(n_r'')$. Furthermore, $A_r'' f_r'(n_r'') > A_r' f_r'(n_r'') \geq A_r' f_r'(n_r') \geq A_v' f_v'(n_v')$ for all v = 1 . . R. The first inequality follows from $A_r'' > A_r'$, the second from $n_r'' \leq n_r'$, and the last from $n_r' > 0$. The preceding series of inequalities and $A_v'' = A_v'$ for $v \neq r$ shows that $A_v' f_v'(n_v'') > A_v' f_v'(n_v')$ for all $v \neq r$ such that $n_v'' > 0$ and hence $n_v'' < n_v'$ for all $v \neq r$ such that $n_v'' > 0$. There must be at least one such $v \neq r$ since otherwise $n_r'' = N > n_r'$ due to $n_s' > 0$, but we have supposed $n_r'' \leq n_r'$. Clearly all $v \neq r$ with $n_v'' = 0$ have $n_v'' \leq n_v'$. Summing over all resources gives $N'' < N'$ because there is at least one $v \neq r$ with $n_v'' < n_v'$. This contradicts the fact that N is constant. Therefore $n_r'' > n_r'$. Next suppose $n_s'' \geq n_s'$. Consider $v \neq s$ and $v \neq r$. For all such v, $A_v''f_v'(n_v'') \leq A_s''f_r'(n_s'') \leq A_s''f_s'(n_s') = A_s'f_s'(n_s')$. Moreover, if $n_v' > 0$ we have $A_s'f_s'(n_s') = A_v'f_v'(n_v')$. This and $A_v'' = A_v'$ implies that for any $v \neq s$ and $v \neq r$ with $n_v' > 0$, it must be true that $n_v'' \geq n_v'$. Clearly the same inequality holds when $n_v' = 0$. Since $n_r'' > n_r'$ and we have supposed $n_s'' \geq n_s'$, summing over resources gives $N'' > N'$. This contradicts the fact that N is constant. Therefore $n_s'' < n_s'$.

(c) Fix $A > 0$. Choose any $N' \neq N''$ and $\mu \in (0, 1)$. Let n' be optimal for (A, N') and let n'' be optimal for (A, N''). Define $n^\mu = \mu n' + (1-\mu)n'' \geq 0$. This is a feasible allocation for the total population $N^\mu = \mu N' + (1-\mu)N''$. It follows that $H(A, N^\mu) \geq \sum A_r f_r(n_r^\mu) = \sum A_r f_r[\mu n_r' + (1-\mu)n_r''] > \sum A_r[\mu f_r(n_r') + (1-\mu)f_r(n_r'')] = \mu H(A, N') + (1-\mu)H(A, N'')$. The strict inequality occurs because due to the strict concavity of f_r we have $f_r[\mu n_r' + (1-\mu)n_r''] > \mu f_r(n_r') + (1-\mu)f_r(n_r'')$ whenever $n_r' \neq n_r''$, and the latter inequality must hold for at least one r because $N' \neq N''$. This establishes the strict concavity of $H(A, N)$ in N. Due to $H(A, 0) = 0$, strict concavity of H gives $H(A, \mu N) > \mu H(A, N)$ for all $N > 0$ and $\mu \in (0, 1)$. This implies $H(A, \mu N)/\mu N > H(A, N)/N$ for all $N > 0$ and $\mu \in (0, 1)$. Thus, $y(A, N) \equiv H(A, N)/N$ is decreasing in N.

(d) Fix $A > 0$ and consider the (unique) optimal allocation $n(A, N)$. We first show that $\lim_{N\to\infty} n_s(A, N) = \infty$ must hold for some s. Suppose instead that for every r there is a finite upper bound $\underline{n}_r$ such that $n_r(A, N) \leq \underline{n}_r$ for all N. Then for any N >

$\sum \underline{n}_r$ we have $\sum n_r(A, N) < N$, which contradicts optimality. Thus, there is some s such that $\lim_{N\to\infty} n_s(A, N) = \infty$. From the assumption that $f_r'(n_r) \to 0$ as $n_r \to \infty$ for r = 1 . . R we obtain $\lim_{N\to\infty} A_s f_s'[n_s(A, N)] = 0$. Next, define $m(A, N) = \max \{A_r f_r'[n_r(A, N)]\}$. There is some $\underline{N}$ such that $N > \underline{N}$ implies $n_s(A, N) > 0$. From the first order conditions for (3.2) this implies $m(A, N) = A_s f_s'[n_s(A, N)]$ for all $N > \underline{N}$. Hence, $\lim_{N\to\infty} m(A, N) = 0$, which implies $\lim_{N\to\infty} A_r f_r'[n_r(A, N)] = 0$ for all r = 1 . . R. Thus, $\lim_{N\to\infty} n_r(A, N) = \infty$ for all r = 1 . . R. From part (c), H(A, N)/N is decreasing in N. Suppose that this ratio has a lower bound $\delta > 0$. This implies $\sum \{A_r f_r[n_r(A, N)] - \delta n_r(A, N)\} \geq 0$. However, we have $f_r(n_r)/n_r \to 0$ as $n_r \to \infty$ for r = 1 . . R. This is obvious if there is a finite upper bound on $f_r(n_r)$. If $f_r(n_r)$ is unbounded then using $f_r'(n_r) \to 0$ as $n_r \to \infty$ gives the same result. The facts that $\lim_{N\to\infty} n_r(A, N) = \infty$ and $f_r(n_r)/n_r \to 0$ as $n_r \to \infty$ for all r = 1 . . R together imply that there is some sufficiently large N such that $A_r f_r[n_r(A, N)] - \delta n_r(A, N) < 0$ for all r = 1 . . R. This contradicts the earlier inequality and gives the desired result $\lim_{N\to\infty} H(A, N)/N = 0$.

(e) Fix $A > 0$. By the envelope theorem H(A, N) is differentiable in N and $H_N(A, N)$ is the Lagrange multiplier for (3.2). Since $H(A, 0) = 0$, $\lim_{N\to 0} H(A, N)/N = \lim_{N\to 0} H_N(A, N)$. When $N > 0$, the first order conditions for (3.2) give $H_N(A, N) = \max \{A_r f_r'[n_r(A, N)]\}$. Because $n_r(A, N)$ is continuous in N with $n_r(A, 0) = 0$ for all r, we have $\lim_{N\to 0} H_N(A, N) = \max \{A_r f_r'(0)\}$.

Proof of Proposition 3.3 (long-run equilibrium).

(a) Suppose max $\{A_r f_r'(0)\} > y^*$. From Proposition 3.2(e), this implies $y(A, 0) > y^*$. From Proposition 3.2(c) and 3.2(d), $y(A, N)$ is decreasing in N and goes to zero as N goes to infinity. By continuity and monotonicity, there is a unique $N(A) > 0$ such that $y[A, N(A)] = y^*$. The population N(A) satisfies $N(A) = \rho[y(A, N(A))]N(A)$ in D3.2 because $\rho(y^*) = 1$. The SRE labor allocation n[A, N(A)] is obtained from (3.2). The null population $N = 0$ with the SRE labor allocation $n = 0$ constitutes an LRE because this satisfies $N = \rho[y(A, N)]N$ from D3.2 and $n = 0$ is optimal for $N = 0$ (in fact, it is the only feasible allocation).

(b) Suppose $A_r f_r'(0) \leq y^*$ for all resources r = 1 . . R. From Proposition 3.2(e), $y(A, 0) \leq y^*$. From Proposition 3.2(c), y(A, N) is decreasing in N. Thus, $y(A, N) < y^*$ for all $N > 0$ and there is no $N(A) > 0$ with $N(A) = \rho[y(A, N(A))]N(A)$. This shows that there is no LRE with $N(A) > 0$. As in part (a), $N = 0$ with $n = 0$ is an LRE.

Proof of Remark.

(a) Suppose there is an LRE with $N(A) > 0$. Note that $y(A, N)$ is decreasing globally with $y[A, N(A)] = y^*$. Use $N(A)$ for N^* in A3.2. Monotone population adjustment ensures $\{N^t\} \to N(A)$ for every initial population $N^0 > 0$. Hence, $N(A)$ is globally asymptotically stable. The null LRE with $N = 0$ is unstable because $\{N^t\} \to N(A) > 0$ for every $N^0 \in (0, N(A))$.

(b) Suppose there is no LRE with $N(A) > 0$. As in the proof of Proposition 3.3(b), $y^* \geq y(A, 0) > y(A, N)$ for all $N > 0$. A straightforward extension of A3.2 gives $\{N^t\} \to 0$ for all $N^0 > 0$ so the LRE with $N = 0$ is globally asymptotically stable.

Proof of Proposition 3.4 (very-long-run equilibrium).

Before addressing (a) and (b), we establish preliminary results. Let (K', N', n') be any non-null VLRE of type S. Write the associated productivities as $A' \equiv A(a, K')$. From the definition of type S we have $k_r' \neq k_r^*$ for all $r \notin S$. From the definition of VLRE in D3.3 we have $n_r' = 0$ for all $r \notin S$.

From D3.3, n' is an optimal time allocation given (K', N'). From the first order conditions for (3.2), we have $H_N(A', N') \geq A_r^* f_r'(n_r')$ for all $r \in S$ where $H_N(A', N')$ is the Lagrange multiplier in the optimization problem for (3.2). Equality must hold for at least one $r \in S$ with $n_r' > 0$ because $N' > 0$ and $n_r' = 0$ for all $r \notin S$. Moreover, $H_N(A', N') \geq A_r(k_r')f_r'(0)$ for all $r \notin S$ by the first order conditions because $n_r' = 0$ is optimal for $r \notin S$.

From D3.3, N' is a stationary population, and $N' > 0$ because the VLRE is non-null. Using Proposition 3.3(a), this implies $H(A', N')/N' = y(A', N') = y^*$.

Now consider a constrained version of problem (3.2) where we maintain $n_r \equiv 0$ for all $r \notin S$ regardless of the population level N. These equalities must hold for any VLRE of type S. This makes the productivities A_r for $r \notin S$ irrelevant. We continue to use the productivities A_r^* for $r \in S$ because these apply to all VLREs of type S. Using the same method of proof as in Proposition 3.2(c), it can be shown that the function $y(A, N)$ for the constrained problem is decreasing in N. No non-null VLRE of type S can have a level of population different from $N' > 0$, because this is the only population that yields food per capita y^* in the constrained problem. The uniqueness of N' implies the uniqueness of $n' = n(A', N')$ for all non-null VLREs of type S.

Next, we show that if there is any non-null VLRE of type S, then (K^S, N^S, n^S) as defined in the text is also a non-null VLRE of type S. As above, let (K', N', n') be a non-null VLRE of type S. We already know that any other non-null VLRE of type S has the same values for N' and n' so it suffices to consider the case $N^S = N'$ and $n^S = n'$. From the definition of type S, we have $k_r^S = k_r' = k_r^*$ for all $r \in S$, which implies $A_r^S = A_r' = A_r^*$ for all $r \in S$. Thus, the two arrays differ at most with respect to the techniques for $r \notin S$ and the associated productivities. From the construction of k_r^S for $r \notin S$ in the text, we have $A_r^S \leq A_r'$ for all $r \notin S$. The fact that n' satisfies the first order conditions from (3.2) for (A', N') implies that n^S satisfies the first order conditions from (3.2) for (A^S, N^S). The Lagrange multipliers in the two problems are identical so $H_N(A', N') = H_N(A^S, N^S)$. This establishes that n^S is an optimal time allocation for (A^S, N^S) as required for a VLRE in D3.3. We have $y(A', N') = y^*$ from Proposition 3.3(a). Because $A_r^S = A_r'$ for all r with $n_r^S = n_r' > 0$, we also have $y(A^S, N^S) = y^*$. Thus, $N^S > 0$ is a stationary population as required for a VLRE in D3.3. Finally, $n_r^S = n_r' = 0$ for all $r \notin S$ implies $k_r^S = k_r^*$ for all r such that $n_r^S > 0$. Thus, K^S is a stationary technology as required for a VLRE in D3.3. Therefore, (K^S, N^S, n^S) is a non-null VLRE of type S.

Suppose condition (a) in Proposition 3.4 does not hold but (K^S, N^S, n^S) is a non-null VLRE of type S. We must have $n_r^S = 0$ for all $r \notin S$. Consider the constrained version of problem (3.2) where the equalities $n_r \equiv 0$ for all $r \notin S$ hold as identities. From the fact that (a) does not hold in Proposition 3.4, the proof used for Proposition 3.3(b) establishes that in the constrained problem there is no $N^S > 0$ with $y(A^S, N^S) = y^*$. This result contradicts the requirement of a stationary population in D3.3. It follows that (K^S, N^S, n^S) cannot be a non-null VLRE of type S. Hence, condition (a) in Proposition 3.4 is necessary. Moreover, if (a) fails, there cannot be any other non-null VLRE of type S because this would falsely imply that (K^S, N^S, n^S) is a non-null VLRE of type S.

Suppose condition (a) in Proposition 3.4 holds. Assume (K^S, N^S, n^S) is a non-null VLRE of type S. We must have $n_r^S = 0$ for all $r \notin S$. The first order conditions for (3.2) imply $H_N(A^S, N^S) \geq A_r(k_r^{min})f_r'(0)$ for all $r \notin S$ by the construction of K^S and the fact that $H_N(A^S, N^S)$ is the Lagrange multiplier from problem (3.2). This confirms that condition (b) in Proposition 3.4 is necessary. Moreover, if (a) holds in Proposition 3.4 but (b) fails, there cannot be any other non-null VLRE of type S, because this would imply that (K^S, N^S, n^S) is a non-null VLRE of type S, which would falsely imply that (b) holds.

Finally, we show sufficiency. Suppose conditions (a) and (b) in Proposition 3.4 both hold. Let $A^S = A(a, K^S)$. By definition, $n^S = n(A^S, N^S)$ solves problem (3.2) for (A^S, N^S) so requirement (c) in D3.3 is satisfied. From condition (b) in Proposition 3.4, the definition of the techniques k_r^S for $r \notin S$, and the fact that $H_N(A^S, N^S)$ is the Lagrange multiplier from problem (3.2), optimality implies $n_r^S = 0$ for all $r \notin S$. This result and the definition of the techniques k_r^S for $r \in S$ imply that requirement (a) in D3.3 is satisfied. From condition (a) in Proposition 3.4, the method of proof for Proposition 3.3(a) can be used to show that when the equalities $n_r = 0$ hold for all $r \notin S$, the population level $N^S = N(A^S)$ is positive. Thus requirement (b) in D3.3 is satisfied and (K^S, N^S, n^S) is non-null.

We have previously shown that if any non-null VLRE of type S exists, the array (K^S, N^S, n^S) is a non-null VLRE of type S. We have also shown that all non-null VLREs of type S have the same population and the same labor allocation. Thus, if any other such VLRE exists, it must have the population $N^S > 0$ and the labor allocation n^S. This ends the proof of Proposition 3.4.

Proof of Proposition 3.5 (convergence to very-long-run equilibrium).

Proposition 3.1 gives transition probabilities for the number of correct digits q_r^{t+1} in the string k_r^{t+1} conditional on the number of correct digits q_r^t in the string k_r^t. Here we must consider transition probabilities for the repertoire K^{t+1} conditional on the repertoire K^t. If resource r is latent ($n_r^t = 0$) then $k_r^{t+1} = k_r^t$ with certainty. If resource r is active ($n_r^t > 0$) and $k_r^t = k_r^*$ then $k_r^{t+1} = k_r^t = k_r^*$ with certainty. If resource r is active ($n_r^t > 0$) and $k_r^t \neq k_r^*$, from Proposition 3.1 the probability that the number of correct digits in k_r stays unchanged is $\exp[-\lambda n_r^t \rho^t (Q-q_r^t)]$ and the probability of one more correct digit is $1 - \exp[-\lambda n_r^t \rho^t (Q-q_r^t)]$. As shown in the proof of Proposition 3.1, we can ignore cases where more than one digit changes simultaneously for the same resource or the number of correct digits decreases. Hence, conditional on k_r^t the probability that a particular incorrect digit becomes correct while all the other digits stay the same is $\{1 - \exp[-\lambda n_r^t \rho^t (Q-q_r^t)]\}/(Q-q_r^t)$. The probability that no digit changes is $\exp[-\lambda n_r^t \rho^t (Q-q_r^t)]$. In the latter case $k_r^{t+1} = k_r^t$. All other transition probabilities for k_r^{t+1} are zero. The updating processes are independent across resources r = 1 . . R given the current repertoire K^t, the current population N^t, and the optimal time allocation $n^t = n(A^t, N^t)$ where $A^t = A(K^t)$. Moreover, $\rho^t = N^{t+1}/N^t$ where N^{t+1} is given by (3.3). Thus, the updated population N^{t+1} and all transition probabilities for K^{t+1} are determined by (K^t, N^t).

(a) Consider a fixed sample path $\{K^t, N^t\}$ for $t \geq 0$. There are finitely many distinct repertoires, so at least one repertoire K' must be repeated infinitely many times. Because no correct digit can ever become incorrect, it is impossible to return to an earlier repertoire after departing from it. Therefore, only one repertoire can occur infinitely many times, and the occurrences of this K' must be consecutive. Let T be the first period in which K' occurs. Using $N^0 > 0$ and $\rho(y) > 0$ for all $y > 0$, (3.3) implies $N^T > 0$. From $K^t = K'$ for $t \geq T$ and constancy of the productivities $A' = A(K')$ for $t \geq T$, A3.2 in section 3.7 implies $\{N^t\} \to N' = N(A')$. The result $\{n^t\} \to n' = n(A', N')$ follows from the continuity of the solutions in (3.2).

(b) Fix the climate $a > 0$. We write the productivity vector for a given repertoire as A(K). When Proposition 3.3(a) applies, we will abbreviate the LRE population as $N[A(K)] = N(K) > 0$ and say that K is non-null. When Proposition 3.3(b) applies, we will write $N[A(K)] = N(K) = 0$ and say that K is null.

From (3.3), if $N^0 = 0$ then $N^t = 0$ for all $t \geq 0$. This implies that the labor allocation is $n^t = 0$ for all $t \geq 0$. Because all resources are latent in all periods, the initial repertoire K^0 is preserved with certainty and $K^t = K^0$ for all $t \geq 0$. Thus, the initial (and terminal) repertoire generates a trivial VLRE. We instead assume $N^0 > 0$, and use $\rho(y) > 0$ for all $y > 0$ in (3.3) to ensure $N^t > 0$ for all $t \geq 0$, whatever the sample path $\{K^t, N^t\}$ may be. This does not rule out the possibility that $\{N^t\} \to 0$ so the population is extinct in the limit. This occurs if the terminal repertoire cannot support food per capita above y^* at any $N > 0$, as in Proposition 3.3(b).

We will say K is of type S when $k_r = k_r^*$ for $r \in S$ and $k_r \neq k_r^*$ for $r \notin S$. When $r \notin S$ is non-empty, we define $m(K) \equiv \max \{a_r g_r(k_r) f_r'(0) \text{ for } r \notin S\}$ to be the maximum marginal product, evaluated at zero labor input, among all the resources whose technique can be improved. For any K where S is a non-empty proper subset of $\{1 \,..\, R\}$ we have

$$H_N[A(K), 0] = \max \{a_r g_r(k_r) f_r'(0) \text{ for } r = 1 \,..\, R\}$$

$\geq \max \{a_r g_r(k_r) f_r'(0) \text{ for } r \notin S\} \equiv m(K)$

$> 0 = H_N[A(K), \infty]$

where the first equality is from Proposition 3.2(e); the weak inequality is obvious; the strict inequality follows from $a > 0$, $g_r(k_r) > 0$ for all r and k_r, and $f_r'(0) > 0$ for all r from A3.1; and the second equality is from Proposition 3.2(d). Define M(K) implicitly by $H_N[A(K), M(K)] \equiv m(K)$. A unique $M(K) \in [0, \infty)$ exists by the above results and the fact that $H_N[A(K), N]$ is continuous and decreasing in N.

Whenever $0 < N \leq M(K)$ we have $H_N[A(K), N] \geq m(K)$ where $H_N[A(K), N]$ is the Lagrange multiplier in problem (3.2). By the first order conditions, this multiplier is equal to the marginal products of all the active resources (those $r \in S$ with $n_r > 0$). Optimality requires $n_r = 0$ for all $r \notin S$, so no improvable technique is used. Whenever $N > M(K)$ we have $H_N[A(K), N] < m(K)$. In this situation the Lagrange multiplier is less than the marginal product evaluated at zero for some $r \notin S$. Hence, optimality requires $n_r > 0$ for at least one $r \notin S$, so some improvable technique is used.

In the special case where S is empty, so all techniques are improvable, we have $m(K) = H_N[A(K), 0] = \max \{a_r g_r(k_r) f_r'(0) \text{ for } r = 1 \,.\,.\, R\}$. This implies $M(K) = 0$. For any $N > 0$ there must be some r with $n_r > 0$, so at least one improvable technique is used. In the special case where $S = \{1 \,.\,.\, R\}$, we have $K = K^*$ and $m(K^*)$ is undefined. For this situation we will define $M(K^*) = \infty$. If K^* is ever reached, it is terminal, with $\{N^t\} \rightarrow N^*$ and $\{n^t\} \rightarrow n(A^*, N^*)$.

Each K, including the two special cases just discussed, belongs to one of two sets:

(i) repertoires with $0 \leq N(K) \leq M(K)$

(ii) repertoires with $0 \leq M(K) < N(K)$

First, consider any K' in set (i). We will show that if K' is terminal then the array (K', N', n') generated by K' is a VLRE. If $N(K') > 0$ then $N' = N(K')$ is an LRE population; $n' = n[A(K'), N')]$ is optimal for the parameters (A', N'); and $N(K') \leq M(K')$ implies $n_r' = 0$ for all $r \notin S$ so no improvable technique is used. Thus, we have a (non-null) VLRE. If $N(K') = 0$ then zero is an LRE population; $n' = 0$ is optimal (in fact, it is the only feasible time allocation); and again no improvable technique is used. Thus, we have a (null) VLRE.

Next, consider any K' in the set (ii). To keep notation compact, write $M' = M(K')$ and $N' = N(K')$ so $0 \leq M' < N'$. We will show that this repertoire cannot be terminal for any initial population $N^0 > 0$. For simplicity, suppose $K^t = K'$ for all $t \geq 0$ (the proof is the same if the time of first occurrence for K' is $T > 0$). Suppose $N^0 \in (M', \infty)$. (Note that if $M' = 0$, this must be true.) A3.2 ensures that if $N^0 < N'$ then $N^t \geq N^0$ for all $t \geq 0$, and if $N' \leq N^0$ then $N^t \geq N'$ for all $t \geq 0$. Thus $M' < \min \{N', N^0\} \leq N^t$ for all $t \geq 0$. Now suppose instead that $N^0 \in (0, M']$. In this case $K^t = K'$ for all $t \geq 0$, $N^0 \leq M' < N'$, and A3.2 together imply that there is a finite $T > 0$ such that $N^T \in (M', \infty)$. This reduces to the previous case with $t = T$ substituted for $t = 0$, so $M' < \min \{N', N^T\} \leq N^t$ for all $t \geq T$.

Recall that for any $N > M'$ some $r \notin S$ must be active. Moreover, from Proposition 3.2(a) the scale effect ensures that if $n_r > 0$ for some N, we have $n_r > 0$ for all larger N for the same resource $r \notin S$. We have shown that for any $N^0 > 0$, there is some finite $T \geq 0$ such that $M' < \min\{N', N^T\} \leq N^t$ for all $t \geq T$. The fact that the productivity vector A' remains constant for $t \geq T$, the scale effect from Proposition 3.2(a), and the construction of M' ensure that for some $r \notin S$ there is a lower bound $\underline{n}_r$ where $0 < \underline{n}_r \leq n_r^t$ for all $t \geq T$. Let $\rho^t \equiv N^{t+1}/N^t$ as in Proposition 3.1. A3.2 guarantees that if $N^T \leq N'$ then $\rho^t \geq \underline{\rho} \equiv 1$ for all $t \geq T$ and if $N^T > N'$ then $\rho^t \geq \underline{\rho} \equiv N'/N^T > 0$ for all $t \geq T$. We also have $Q - q_r^t \geq 1$ for all $t \geq T$ due to $k_r' \neq k_r^*$. Using these lower bounds in Proposition 3.1, for each $t \geq T$ the probability that k_r remains constant cannot exceed $\exp(-\lambda\underline{\rho}\underline{n}_r) < 1$. Over the unbounded interval $t \geq T$ the probability that k_r is constant vanishes. Therefore, K' cannot be terminal.

We have shown in Proposition 3.5(a) that for any specified initial conditions (K^0, N^0), each sample path has some terminal repertoire. There are finitely many repertoires, so we can partition the set of sample paths into finitely many subsets distinguished by the terminal repertoire K'. We have also shown that when $N^0 > 0$, any terminal repertoire such that $0 \leq M(K') < N(K')$ has probability zero. This implies that the terminal repertoire has $0 \leq N(K') \leq M(K')$ with probability one. Finally, we have shown that if the latter inequalities hold, the terminal repertoire K' generates a terminal array (K', N', n') that is a VLRE.

(c) Suppose $0 < N^t < N[A(K^t)]$. From A3.2, Proposition 3.1, conservation of latent strings, and the fact that $N(A)$ is non-decreasing we have $N^t < N^{t+1} < N[A(K^t)] \leq N[A(K^{t+1})]$. When $0 < N^0 < N[A(K^0)]$, we can repeat the argument to obtain $0 < N^t < N^{t+1}$ for all $t \geq 0$. When $0 < N^0 = N[A(K^0)]$, we have $N^t = N[A(K^0)]$ for $0 \leq t \leq T$ where T is the first period (if any) in which a mutation occurs for an active resource. This yields $N^0 = N^T < N[A(K^T)]$. For $t \geq T$, $\{N^t\}$ is increasing as before. When $N^0 > N[A(K^0)] \geq 0$, A3.2 ensures $N^0 > N^1 > N[A(K^0)]$. If there is a period $T \geq 1$ in which a mutation to an active resource yields $N^T \leq N[A(K^T)]$ then $\{N^t\}$ is non-decreasing for $t \geq T$ by the reasoning used above (and increasing if $N^T < N[A(K^T)]$ holds). Otherwise, we have $N^t > N[A(K^t)]$ for all $t \geq 0$. From A3.2 this implies $N^t > N^{t+1} > N[A(K^t)]$ for all $t \geq 0$ and $\{N^t\}$ is decreasing.

Proof of Proposition 3.6 (neutral shocks).

(a) Let the climate change permanently to a' at the start of period $t = 0$ before labor is allocated. The repertoire and population (K^0, N^0) at $t = 0$ are inherited from the previous VLRE. The productivity vector in period $t = 0$ is $\theta A^0 = \theta A(a^0, K^0) = A(\theta a^0, K^0)$. Due to the neutrality of the shock, the optimal labor allocation in (3.2) for $t = 0$ is unaffected (the solution $n(A, N)$ is homogeneous of degree zero in A). We use n^0 interchangeably for the labor allocation in the original VLRE and the allocation in period $t = 0$ after the climate change occurs. If K^0 is held constant over all subsequent periods, the new LRE population is $N' = N(\theta A^0) > 0$.

We first show that for any $t \geq 0$, if $K^t = K^0$ and $N^t \in (N', N^0]$, then $K^{t+1} = K^0$ and $N^{t+1} \in (N', N^0]$. Using $K^t = K^0$, the labor allocation associated with (K^t, N^t) is $n^t = n(\theta A^0, N^t)$. Two necessary conditions for $k_r^{t+1} \neq k_r^t$ are (i) $k_r^t \neq k_r^*$ and (ii) $n_r^t > 0$. Any r satisfying (ii) has $n_r(\theta A^0, N^t) > 0$. Using $N^0 \geq N^t$ and Proposition 3.2(a) this implies $n_r(\theta A^0, N^0) > 0$. But then $n_r(A^0, N^0) > 0$ by the homogeneity of $n(A, N)$ in A. This implies $k_r^0 = k_r^t = k_r^*$ because (K^0, N^0, n^0) is a VLRE for climate a^0. Thus (i) cannot hold. This shows that $k_r^{t+1} = k_r^t$ for all r and hence $K^{t+1} = K^0$. We also have $H(\theta A^0, N')/N' = y^* > H(\theta A^0, N^t)/N^t$ because $N' > 0$ is the LRE population for θA^0, $N^t > N'$, and food per capita is decreasing in N for fixed productivity levels. The assumption $N' > 0$ along with A3.2 implies $N^{t+1} \in (N', N^t) \subseteq (N', N^0]$.

Clearly $K^t = K^0$ and $N^t \in (N', N^0]$ for $t = 0$. We have also shown that for any $t \geq 0$, if $K^t = K^0$ and $N^t \in (N', N^0]$ then $K^{t+1} = K^0$ and $N^{t+1} \in (N', N^0]$.

Together these imply $K^t = K^0$ for all $t \geq 0$. Moreover, A3.2 implies $\{N^t\} \rightarrow N' = N(\theta A^0) < N^0$. This establishes deterministic convergence to the unique VLRE (K', N', n') such that $K' = K^0$, $N' = N(\theta A^0) < N^0$, and $n' = n[\theta A^0, N(\theta A^0)]$.

Suppose $n_r^0 = n_r(A^0, N^0) = 0$. If $n_r' = n_r(\theta A^0, N') > 0$ then $n_r(A^0, N') > 0$ by homogeneity. Moreover, $n_r(A^0, N^0) > 0$ from $N' < N^0$ and Proposition 3.2(a). This is a contradiction, and therefore $n_r^0 = 0$ implies $n_r' = 0$. This shows that the active resources in n' are a subset of the active resources in n^0.

Starting from the non-null VLRE (K', N', n') associated with climate a', suppose in period $t = 0$ the climate returns permanently to $a^0 = a'/\theta$. We will show that for any $t \geq 0$, if $K^t = K'$ and $N^t \in [N', N^0)$, then $K^{t+1} = K'$ and $N^{t+1} \in [N', N^0)$. Using $K^t = K' = K^0$, the labor allocation associated with (K^t, N^t) is $n^t = n(A^0, N^t)$. As before, two necessary conditions for $k_r^{t+1} \neq k_r^t$ are (i) $k_r^t \neq k_r^*$ and (ii) $n_r^t > 0$. Suppose there is some r satisfying (ii). For this r we have $n_r(A^0, N^t) > 0$. Using $N^t < N^0$ and Proposition 3.2(a) implies $n_r(A^0, N^0) > 0$. But each r with $n_r^0 = n_r(A^0, N^0) > 0$ has $k_r^0 = k_r^t = k_r^*$ because (K^0, N^0, n^0) is a VLRE for a^0 and $K^t = K' = K^0$. Thus, (i) cannot hold. This shows that $k_r^{t+1} = k_r^t$ for all r and therefore $K^{t+1} = K'$. Using $N^t < N^0$, we have $H(A^0, N^0)/N^0 = y^* < H(A^0, N^t)/N^t$ because (K^0, N^0, n^0) is a VLRE for a^0 and food per capita is decreasing in N for fixed productivities. Assumption A3.2 then implies $N^{t+1} \in [N^t, N^0) \subseteq [N', N^0)$. This yields the desired result. Clearly $K^t = K'$ and $N^t \in [N', N^0)$ for $t = 0$ (to avoid notational confusion recall that in this paragraph, the system is starting from (K', N', n') and N^0 is the LRE population for a different VLRE). We have shown that for any $t \geq 0$, if $K^t =$

K' and $N^t \in [N', N^0)$, then $K^{t+1} = K'$ and $N^{t+1} \in [N', N^0)$. This implies that $K^t = K'$ $= K^0$ for all $t \geq 0$. Also, A3.2 implies that $\{N^t\} \to N^0$. This shows deterministic convergence to the original VLRE (K^0, N^0, n^0).

(b) By Proposition 3.5(b), the terminal array (K', N', n') is a VLRE. From the fact that (K^0, N^0, n^0) is a VLRE for climate a^0 and $H(A, N)$ is linearly homogeneous in A, we have $y^* = H(A^0, N^0)/N^0 < H(\theta A^0, N^0)/N^0$. This shows that $N^0 < N(\theta A^0)$.

Necessity. Suppose $n_r[\theta A^0, N(\theta A^0)] = 0$ for all r such that $k_r^0 \neq k_r^*$. Choose any $t \geq 0$. We will show that if $K^t = K^0$ and $N^t \in [N^0, N(\theta A^0))$, then $K^{t+1} = K^0$ and $N^{t+1} \in [N^0, N(\theta A^0))$. Consider any resource r. Two necessary conditions for $k_r^{t+1} \neq k_r^t$ are (i) $k_r^t \neq k_r^*$ and (ii) $n_r^t > 0$. Using $N^t < N(\theta A^0)$ and Proposition 3.2(a), (ii) gives $n_r[\theta A^0, N(\theta A^0)] > 0$. But our initial supposition that $n_r[\theta A^0, N(\theta A^0)] = 0$ for all r such that $k_r^0 \neq k_r^*$ implies $k_r^0 = k_r^*$. Using $K^t = K^0$, this implies $k_r^t = k_r^0 = k_r^*$, so (i) cannot hold. Therefore, $k_r^{t+1} = k_r^t$ for all r and $K^{t+1} = K^t = K^0$. Using $N^t < N(\theta A^0)$, the definition of $N(\theta A^0)$, and the fact that food per capita is decreasing in N, we have $H[\theta A^0, N(\theta A^0)]/N(\theta A^0) = y^* < H(\theta A^0, N^t)/N^t$. Using $N^0 \leq N^t$, A3.2 yields $N^{t+1} \in [N^t, N(\theta A^0)) \subseteq [N^0, N(\theta A^0))$ as claimed. Clearly $K^t = K^0$ and $N^t \in [N^0, N(\theta A^0))$ for $t = 0$, so $K^t = K^0$ for all $t \geq 0$. Hence, $K' = K^0$ and $N' = N(\theta A^0)$. This shows that a necessary condition for $K' \neq K^0$ is $n_r[\theta A^0, N(\theta A^0)] > 0$ for some r such that $k_r^0 \neq k_r^*$.

Sufficiency. Suppose $n_r[\theta A^0, N(\theta A^0)] > 0$ for some r such that $k_r^0 \neq k_r^*$, but $K' = K^0$. Because $K' = K^0$, the terminal productivity vector is $A' = \theta A^0$. Furthermore, because (K', N', n') is a VLRE we have $H(\theta A^0, N')/N' = y^*$ where $N' = N(\theta A^0)$.

The conditions for VLRE also require $n_r' = n_r[\theta A^0, N(\theta A^0)] = 0$ for all r such that $k_r' \neq k_r^*$. From $K' = K^0$ this is true for all r such that $k_r^0 \neq k_r^*$. But by assumption we have $n_r[\theta A^0, N(\theta A^0)] > 0$ for some r such that $k_r^0 \neq k_r^*$. This is a contradiction and therefore $K' \neq K^0$.

We have shown that $N(\theta A^0) > N^0$ holds in all cases. Moreover, we have shown that if (*) does not hold then $N' = N(\theta A^0)$. Assume (*) does hold. We need to show that $N' > N(\theta A^0)$. Define $A' \equiv A(a', K')$. Because (K', N', n') is a VLRE for a' we have $H(A', N')/N' = y^* = H[\theta A^0, N(\theta A^0)]/N(\theta A^0)$. We cannot have $N' < N(\theta A^0)$ because then $A' \geq \theta A^0$ gives $y^* = H[\theta A^0, N(\theta A^0)]/N(\theta A^0) < H(\theta A^0, N')/N' \leq H(A', N')/N' = y^*$, which is a contradiction. Thus, $N' \geq N(\theta A^0)$.

Next, we rule out $N' = N(\theta A^0)$. Suppose $N' = N(\theta A^0)$ holds. This gives $H(A', N')/N' = y^* = H[\theta A^0, N(\theta A^0)]/N(\theta A^0)$ and so $H(A', N') = H[\theta A^0, N(\theta A^0)]$. Consider the case $n' \neq n[\theta A^0, N(\theta A^0)]$. From the uniqueness of solutions in (3.2) and the fact that both of the labor allocations involved are feasible, we have $H(A', N') = \sum \theta a_r^0 g_r(k_r') f_r(n_r') > \sum \theta a_r^0 g_r(k_r') f_r[n_r(\theta A^0, N')] \geq \sum \theta a_r^0 g_r(k_r^0) f_r[n_r(\theta A^0, N')] = H[\theta A^0, N(\theta A^0)]$ where the weak inequality follows from $g_r(k_r') \geq g_r(k_r^0)$ for all r due to Proposition 3.1. This contradicts $H(A', N') = H[\theta A^0, N(\theta A^0)]$.

Now suppose $N' = N(\theta A^0)$ and consider the case $n' = n[\theta A^0, N(\theta A^0)]$. $H(A', N') = H[\theta A^0, N(\theta A^0)]$ implies that $\sum \theta a_r^0 g_r(k_r') f_r(n_r') = \sum \theta a_r^0 g_r(k_r^0) f_r(n_r')$ or $\sum a_r^0 f_r(n_r')[g_r(k_r') - g_r(k_r^0)] = 0$ where $g_r(k_r') \geq g_r(k_r^0)$ for all r. Because (*) holds with $n' = n[\theta A^0, N(\theta A^0)]$, there exists some r for which $n_r' = n_r[\theta A^0, N(\theta A^0)] > 0$ and $k_r^0 \neq k_r^*$. Because (K', N', n') is a VLRE, $n_r' > 0$ implies $k_r' = k_r^*$. Hence, there is at least one r with $f_r(n_r') > 0$ and $g_r(k_r') > g_r(k_r^0)$. This contradicts $\sum a_r^0 f_r(n_r')[g_r(k_r') - g_r(k_r^0)] = 0$. Therefore, (*) implies $N' > N(\theta A^0)$.

Suppose a^0 is permanently restored and (*) does not hold. We want to show that starting from (K', N', n') the system converges to (K^0, N^0, n^0). We have already shown that if (*) does not hold then $K' = K^0$. The reversion to a^0 from θa^0 is a neutral negative shock. Proposition 3.6(a) shows that the system converges to a VLRE (K'', N'', n'') such that $K'' = K' = K^0$. The productivity vector for this VLRE is $A'' = A(a^0, K'') = A^0$. Furthermore, $H(A^0, N^0)/N^0 = y^*$ from the fact that (K^0, N^0, n^0) is a non-null VLRE. This implies $N'' = N^0$. The uniqueness of the solution in (3.2) gives $n'' = n(A^0, N^0) = n^0$.

Assume a^0 is permanently restored and (*) does hold. We want to show that starting from (K', N', n') the system converges to a VLRE (K'', N'', n'') with $K'' = K' \neq K^0$ and $N' > N'' \geq N^0$. We have already shown that (*) implies $K' \neq K^0$. Proposition 3.6(a) shows that the system converges to a VLRE with $K'' = K'$ and $N' > N''$. Thus, it suffices to show $N'' \geq N^0$, and to establish conditions under which this inequality is strict. First, we show that $N'' \geq N^0$. Let $A'' = A(a^0, K'')$, where $A'' \geq A^0$ due to Proposition 3.1. Because H(A, N) is non-decreasing in A, we have $H(A'', N^0)/N^0 \geq H(A^0, N^0)/N^0 = y^*$. This implies $N'' \geq N^0$.

Now continue to assume (*) does hold and suppose $N'' = N^0$ with $n'' \neq n^0$. Because n^0 is feasible in the allocation problem for (A'', N'') and solutions in (3.2) are unique, $H(A'', N'') = \sum a_r^0 g_r(k_r'')f_r(n_r'') > \sum a_r^0 g_r(k_r'')f_r(n_r^0) \geq \sum a_r^0 g_r(k_r^0)f_r(n_r^0) = H(A^0, N^0)$. This gives $y^* = H(A'', N'')/N'' > H(A^0, N^0)/N^0 = y^*$, which is a contradiction. Therefore, if $N'' = N^0$ then $n'' = n^0$. We have two possibilities: (i) $N'' = N^0$ and $n'' = n^0$ or (ii) $N'' > N^0$ and $n'' \neq n^0$. In either case, because (K^0, N^0, n^0) is a VLRE we have $k_r^0 = k_r^*$ for all r with $n_r^0 > 0$. Proposition 3.1 implies $k_r'' = k_r' = k_r^*$ for all r with $n_r^0 > 0$.

We proceed to consider two situations and will show that they correspond to the possibilities (i) and (ii) above.

(i) Suppose $H_N(A^0, N^0) \geq a_r^0 g_r(k_r'') f_r'(0)$ for all r such that $n_r^0 = 0$. We know $k_r'' = k_r^0$ for all r such that $n_r^0 > 0$. Together these imply that n^0 satisfies the first order conditions for (3.2) with parameters (A'', N^0). The first order conditions for (3.2) are sufficient for a solution so $H(A'', N^0)/N^0 = H(A^0, N^0)/N^0 = y^*$. This shows that ($K''$, N^0, n^0) is a VLRE for the climate a^0. But from Proposition 3.6(a), the VLRE (K'', N'', n'') is unique. Thus, $N'' = N^0$ and $n'' = n[A(a^0, K''), N^0] = n^0$.

(ii) Suppose $H_N(A^0, N^0) < a_r^0 g_r(k_r'') f_r'(0)$ for some r such that $n_r^0 = 0$. In this case n^0 does not satisfy the first order conditions for (3.2) with parameters (A'', N^0). Thus, $n[A(a^0, K''), N^0] \neq n^0$. Using $N'' \geq N^0$ and the uniqueness of solutions in (3.2), this gives $H(A'', N'') \geq H(A'', N^0) > \sum a_r^0 g_r(k_r'') f_r(n_r^0) \geq \sum a_r^0 g_r(k_r^0) f_r(n_r^0) = H(A^0, N^0)$. Now suppose $N'' = N^0$. This implies $y^* = H(A'', N'')/N'' > H(A^0, N^0)/N^0 = y^*$, which is a contradiction. Therefore, $N'' > N^0$. This implies $n'' \neq n^0$.

Finally, continue with the supposition in the preceding paragraph for case (ii). If $n_r'' = 0$ for all r such that $n_r^0 = 0$, then from $N'' > N^0$ there must be at least one s with $0 < n_s^0 < n_s''$. Using $k_r'' = k_r^*$ for all r with $n_r^0 > 0$, strict concavity of f_s from A3.1, the first order conditions for (3.2), and the fact that H_N is the Lagrange multiplier in the first order conditions, this yields $H_N(A'', N'') = a_s^0 g_s(k_s'') f_s'(n_s'')$ $< a_s^0 g_s(k_s^*) f_s'(n_s^0) = H_N(A^0, N^0)$. But then $H_N(A'', N'') < H_N(A^0, N^0) <$ $a_r^0 g_r(k_r'') f_r'(0)$ for some r such that $n_r^0 = 0$. This contradicts the optimality of $n_r'' = 0$ for all r such that $n_r^0 = 0$. Therefore $n_r'' > 0$ for at least one r such that $n_r^0 = 0$. This completes the proof of Proposition 3.6.

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Chapter 4: The Transition to Sedentism

Proofs of Formal Propositions

Proof of Proposition 4.1 (optimal labor allocation).

In all cases the solution in (4.2) is unique due to strict concavity, and first order necessary conditions (FOC) for a maximum are also sufficient.

(a) By A4.1 there is a unique $x_f > 0$ satisfying the condition in part (a) of Proposition 4.1.

(i) We have $f'(L) \geq f'(x_f) \equiv kg'(0)$. Thus, $L_f = L$ and $L_g = 0$ satisfies the FOC for a maximum.

(ii) The solution cannot have $L_f = 0$ because then the FOC implies $kg'(L) \geq f'(0) > kg'(0) > kg'(L)$, which is a contradiction. The solution cannot have $L_g = 0$ because then the FOC implies $f'(L) \geq kg'(0) \equiv f'(x_f) > f'(L)$, which is a contradiction. It follows that $L_f > 0$ and $L_g > 0$. The FOC for an interior solution is $f'(L_f) = kg'(L_g)$.

(b) The proof parallels (i) and (ii) in part (a).

(c) The proof parallels (ii) in part (a).

(d) $H(L, k)$ is continuous in (L, k) by the theorem of the maximum. It is increasing in L for a fixed k because f is increasing in L_f and g is increasing in L_g. To show that H is strictly concave in L, fix $k > 0$, choose any $L' \neq L''$, and choose any $\mu \in (0, 1)$. Let (L_f', L_g') be optimal for the total labor supply L' and let (L_f'', L_g'') be optimal for the total labor supply L''. Define $L_f^* \equiv \mu L_f' + (1-\mu)L_f''$ and $L_g^* \equiv \mu L_g' + (1-\mu)L_g''$. Notice that (L_f^*, L_g^*) is a feasible allocation of the total labor supply $L^* = \mu L' + (1-\mu)L''$. This implies $H(L^*, k) \geq f(L_f^*) + kg(L_g^*) > \mu f(L_f') + (1-\mu)f(L_f'') + \mu kg(L_g') + (1-\mu)kg(L_g'') = \mu H(L', k) + (1-\mu)H(L'', k)$. The strict inequality in this sequence follows because $f(L_f^*) > \mu f(L_f') + (1-\mu)f(L_f'')$ due to the strict concavity of f, and $kg(L_g^*) > \mu kg(L_g') + (1-\mu)kg(L_g'')$ due to the strict concavity of g. This establishes that H(L, k) is strictly concave in L.

(e) Due to H(0, k) = 0, the strict concavity of H implies $H(\mu L, k) > \mu H(L, k)$ for all $L > 0$ and $\mu \in (0, 1)$. This yields $H(\mu L, k)/\mu L > H(L, k)/L$ for all $L > 0$ and $\mu \in (0, 1)$. Thus, $h(L, k) \equiv H(L, k)/L$ is decreasing in L.

(f) (i) Let $[L_f(L), L_g(L)]$ be the optimal allocation for $L > 0$. One and only one of parts (a), (b), or (c) above must apply. Suppose (a) applies. Then for sufficiently small $L > 0$ we have $H(L, k) = f(L)$ and $h(L, k) = f(L)/L$. This implies that as $L \to 0$, we have $h(L, k) \to f'(0) > kg'(0)$. Therefore, $h(0, k) = \max \{f'(0), kg'(0)\}$. The proofs for (b) and (c) are similar.

(ii) For a fixed $k > 0$, whenever L is sufficiently large we have $L_f(L) > 0$ and $L_g(L) > 0$ with $f'[L_f(L)] = kg'[L_g(L)]$. From part (e) above, h(L, k) is decreasing in L. Suppose there is a lower bound $\delta > 0$ such that $h(L, k) \geq \delta$ for all $L > 0$. This implies $f[L_f(L)]/L + kg[L_g(L)]/L \geq \delta > 0$ for all $L > 0$. We have $f(L_f)/L_f \to 0$ as $L_f \to \infty$. This is obvious if f has a finite upper bound. If f is unbounded, then using $f'(L_f) \to 0$ as $L_f \to \infty$ from A4.1 gives the same result. Likewise, $g(L_g)/L_g \to 0$ as $L_g \to \infty$. The lower bound $\delta > 0$ implies that $L_f(L) \to \infty$ and $L_g(L) \to \infty$ cannot both hold when $L \to \infty$. Therefore, one or the other must have a finite upper bound $M > 0$. Suppose $L_f(L) \leq M$ for all L. Then $f'[L_f(L)] \geq f'(M) > 0$ for all $L > 0$. Since $L_f(L)$ has a finite upper bound, $L_g(L) \to \infty$ must hold as $L \to \infty$. Thus, $kg'[L_g(L)] \to 0$ as $L \to \infty$. For sufficiently large L, this contradicts the first order condition $f'[L_f(L)] = kg'[L_g(L)]$. The same is true if $L_g(L)$ has a finite upper bound. Thus, neither $L_f(L)$ nor $L_g(L)$ has an upper bound, so there is no lower

bound $\delta > 0$ and $h(L, k) \to 0$ as $L \to \infty$.

<u>Proof of Proposition 4.2</u> (short-run equilibrium).

(a) For all $y \geq \theta_A h(0)$ we have $\eta(y/\theta_A) = 0$ due to (4.3a). Also, $\theta_A > \theta_B$ implies $y \geq \theta_B h(0)$ so we have $\eta(y/\theta_B) = 0$ due to (4.3b). Hence, $D(y) = 0 < N$ and no such y solves (4.3c). For $y \leq \theta_A h(0)$, D(y) is continuous and decreasing because $\eta(y/\theta_A)$ is continuous and decreasing while $\eta(y/\theta_B)$ is continuous and non-increasing. Also, $D(y) \to \infty$ as $y \to 0$ because $h(L) \to 0$ as $L \to \infty$ from Proposition 4.1(f). Because $N > 0$ is finite, $D[\theta_A h(0)] = 0$, $D(0) = \infty$, and D(y) is continuous and decreasing on $(0, \theta_A h(0)]$, there is a unique $y(N) \in (0, \theta_A h(0))$ such that $D[y(N)] = N$.

(b) Consider the unique value of y from part (a) that solves (4.3c) and choose L_A and L_B as in Proposition 4.2(b). The fact that y solves (4.3c) implies that condition (c) in D4.1 holds. Using $y < \theta_A h(0)$ when $D(y) = N$ as in Proposition 4.2(a), with $L_A = \eta(y/\theta_A) > 0$ as in (4.3a), implies $\theta_A h(L_A) = y$ so that condition (a) in D4.1 holds. From (4.3b), either (i) $L_B = \eta(y/\theta_B) > 0$, which implies $\theta_B h(L_B) = y$, or (ii) $L_B = \eta(y/\theta_B) = 0$, which implies $\theta_B h(0) \leq y$. Condition (b) in D4.1 holds in either case. Because all the requirements in the definition D4.1 are satisfied, (y, L_A, L_B) is an SRE. To show that it is unique, suppose (y', L_A', L_B') is a different SRE. Condition (a) in D4.1 implies $L_A' = \eta(y'/\theta_A) > 0$ and condition (b) in D4.1 implies either (i) $L_B' = \eta(y'/\theta_B) > 0$ or (ii) $L_B' = \eta(y'/\theta_B) = 0$ with $y' \geq \theta_B h(0)$. Condition (c) in D4.1 implies $\lambda L_A' + (1-\lambda)L_B' = N$. This implies $D(y') \equiv \lambda\eta(y'/\theta_A) + (1-\lambda)\eta(y'/\theta_B) = N$. Proposition 4.2(a) states that there is a unique solution to (4.3c), so we have $y = y'$. Thus, $L_A' \neq L_A$ or $L_B' \neq L_B$ or both. But this is impossible because for any given y there is a unique solution for L_A from (a) in D4.1 and a unique solution for L_B from (b) in D4.1.

(c) Continuity of y(N) follows from continuity of D(y) in (4.3c). y(N) is decreasing because D(y) is decreasing over the relevant range. To show that $y(N) \to 0$ as $N \to \infty$, use (4.3c) to write the identity $D[y(N)] \equiv \lambda\eta[y(N)/\theta_A] + (1-\lambda)\eta[y(N)/\theta_B] \equiv N$ where y(N) is decreasing. Suppose there is a lower bound $\delta > 0$ such that $y(N) \geq \delta$ for all $N > 0$. Because η is decreasing, $D[y(N)] \leq \lambda\eta(\delta/\theta_A) + (1-\lambda)\eta(\delta/\theta_B)$ for all $N > 0$. Choosing an N that exceeds the right-hand side of this inequality gives a contradiction. Thus, there is no such lower bound and $y(N) \to 0$ as $N \to \infty$.

Proof of Proposition 4.3 (long-run equilibrium).

(a) Suppose $\theta_A h(0, k) > y^*$. By Proposition 4.1 there is an $L_A > 0$ such that $\theta_A h(L_A, k) = y^*$ or equivalently $L_A = \eta(y^*/\theta_A, k) > 0$. Set $L_B = \eta(y^*/\theta_B, k) \geq 0$ and $N = \lambda L_A + (1-\lambda)L_B > 0$. The triple (L_A, L_B, N) is an LRE because (L_A, L_B, y^*) is an SRE for N. Any other LRE must have the same (L_A, L_B) to satisfy conditions (a) and (b) in D4.1 with food income y^*. It must therefore have the same N to satisfy condition (c) in D4.1. This establishes uniqueness.

(b) Suppose there is some (L_A, L_B, N) with $N > 0$ that is an LRE. From condition (a) in D4.1 we must have $\theta_A h(L_A, k) = y^*$ with $L_A > 0$. However, this implies $\theta_A h(0, k) > y^*$, which contradicts the assumption $\theta_A h(0, k) \leq y^*$.

Proof of Proposition 4.4 (very-long-run equilibrium).

(a) Necessity. From (b) in D4.3, a necessary condition for a VLRE with such a value of k is $L_{Ag} = L_{Bg} = 0$. Using Proposition 4.1, $L_{Ag} = 0$ occurs if and only if $f'(L_A) \geq kg'(0)$. From (a) in D4.3, another necessary condition is that (L_A, L_B, N) form an LRE for the given k. In turn, this requires that (L_A, L_B, y*) form a SRE for N > 0. From (a) in D4.1, this implies $\theta_A h(L_A, k) = y^*$. Due to $L_{Ag} = 0$ this reduces to the condition $\theta_A f(L_A)/L_A = y^*$. There is such an $L_A > 0$ iff $\theta_A f'(0) > y^*$. Together these results show the necessity of the conditions in Proposition 4.4(a).

Sufficiency. Suppose the conditions in Proposition 4.4(a) are satisfied. Compute L_A, L_B, and N as in the Proposition. We need to show that this gives a non-null VLRE. $L_A > 0$ implies N > 0 so any VLRE will be non-null. Condition (b) in D4.3 is satisfied because (i) $f'(L_A) \geq kg'(0)$ implies $L_{Ag} = 0$ from Proposition 4.1; and (ii) $\theta_B < \theta_A$ implies $L_B < L_A$, which implies $f'(L_B) \geq kg'(0)$, and this in turn implies $L_{Bg} = 0$ from Proposition 4.1. Condition (a) in the definition of VLRE from D4.3 is satisfied because the definition of LRE from D4.2 is satisfied.

(b) When k = k*, condition (b) in D4.3 is satisfied. Condition (a) in D4.3 reduces to the conditions for a (non-null) LRE. By Proposition 4.3(a), these conditions can be satisfied iff $\theta_A h(0, k^*) > y^*$. When this is true, condition (a) in D4.1 implies that $L_A > 0$ satisfies $\theta_A h(L_A, k^*) = y^*$. Condition (b) in D4.1 implies that if $\theta_B h(0, k^*) > y^*$ then $L_B > 0$ satisfies $\theta_B h(L_B, k^*) = y^*$; otherwise $L_B = 0$. Condition (c) in D4.1 gives $N = \lambda L_A + (1-\lambda)L_B > 0$.

Proof of Proposition 4.5 (baseline VLRE).

By Proposition 4.4(a), if $k < k^*$ there is a non-null VLRE iff $\theta_A f'(0) > y^*$ and the value of L_A such that $\theta_A f(L_A)/L_A = y^*$ gives $f'(L_A) \geq kg'(0)$. This gives conditions (4.9a) and (4.9c). It is automatic that only hunting is used. Adding the requirement that only sites of type A are active implies $L_B = 0$. By Proposition 4.4(a) this holds iff $\theta_B f'(0) \leq y^*$. This gives condition (4.9b).

Proof of Proposition 4.6 (short run; climate only).

(a) In the baseline equilibrium $L_B^0 = 0$ because type-B sites are not used. Thus, $N^0 = \lambda L_A^0 > 0$ where L_A^0 is the baseline population at a type-A site. We have $f'(L_A^0) \geq k^0 g'(0)$ from (4.9c). Because $N^0 = \lambda L_A + (1-\lambda)L_B$ is fixed in the short run and $L_B \geq 0$ under the new climate regime, we must have $L_A \leq L_A^0$ under the new climate regime. Because there is no change in k^0, we have $f'(L_A) \geq k^0 g'(0)$ so gathering cannot be used at the type-A sites. Because $L_B < L_A$ in every SRE, we have $f'(L_B) > k^0 g'(0)$ so gathering cannot be used at type-B sites either.

(b) In the baseline equilibrium regional population is $N^0 = \lambda L_A^0 = \lambda\eta(y^*/\theta_A^0, k^0)$ where the second equality follows from condition (a) in D4.1 and (4.3a). From (4.5), type-B sites are used in period t = 0 under the new climate regime iff $N^0 > N^*(k^0) \equiv \lambda\eta[\theta_B^* h(0, k^0)/\theta_A^*, k^0]$ or equivalently $\eta(y^*/\theta_A^0, k^0) > \eta[\theta_B^* h(0, k^0)/\theta_A^*, k^0]$. Because $\eta = h^{-1}$ is decreasing in its first argument for a fixed k^0, this holds iff $y^*/\theta_A^0 < \theta_B^* h(0, k^0)/\theta_A^*$. Due to (4.9c) for baseline equilibrium, $h(0, k^0) \equiv \max \{f'(0), k^0 g'(0)\} = f'(0)$. Substituting this into the previous inequality gives the result in Proposition 4.6(b).

Proof of Proposition 4.7 (long run; climate plus population).

First, we prove some preliminary results. Define $L_h > 0$ to satisfy $\theta_A{}^*h(L_h, k^0) \equiv \theta_B{}^*h(0, k^0)$. This L_h exists and is unique due to $\theta_A{}^* > \theta_B{}^*$ and Proposition 4.1(d)-(f). By the definition of SRE from D4.1 we have $L_B = 0$ when $0 \leq L_A \leq L_h$ and $L_B > 0$ when $L_A > L_h$.

Observe from (4.9c) in Proposition 4.5 that $f'(0) > k^0g'(0)$ holds. Therefore, $h(0, k^0) = f'(0)$ from Proposition 4.1(f). Define $x_f^0 > 0$ by $f'(x_f^0) \equiv k^0g'(0)$ as in Proposition 4.1(a). At sites of type A we have $L_{Ag} = 0$ when $0 \leq L_A \leq x_f^0$ and $L_{Ag} > 0$ when $L_A > x_f^0$.

Denote the SRE population at type-A sites by $L_A(N)$, which is continuous and increasing with $L_A(0) = 0$ and $L_A(\infty) = \infty$. These properties follow from Proposition 4.2 and the properties of the inverse function η defined in (4.3). Denote the SRE population at type-B sites by $L_B(N)$.

The properties of $L_A(N)$ imply that there is a unique $N_h > 0$ such that $L_h \equiv L_A(N_h)$. Furthermore, $N \leq N_h$ implies $L_A(N) \leq L_h$ and $L_B(N) = 0$, while $N > N_h$ implies $L_A(N) > L_h$ and $L_B(N) > 0$. Similarly, there is a unique $N_f > 0$ such that $x_f^0 \equiv L_A(N_f)$. Moreover, $N \leq N_f$ implies $L_A(N) \leq x_f^0$ and $L_{Ag}(N) = 0$, while $N > N_f$ implies $L_A(N) > x_f^0$ and $L_{Ag}(N) > 0$.

Assume $k^0 < \underline{k}$ as in Proposition 4.7(a). We want to show that this implies $N_h < N_f$. Suppose instead $N_h \geq N_f$. This implies $L_B(N_f) = 0$ and $L_A(N_f) = N_f/\lambda$. By the construction of N_h we have $\theta_A{}^*h[L_A(N_h), k^0] = \theta_B{}^*f'(0)$. By $N_f \leq N_h$, the fact that $L_A(N)$ is increasing, the fact that h is decreasing, and the earlier result $L_A(N_f) = N_f/\lambda$, we have $\theta_A{}^*h(N_f/\lambda, k^0) \geq \theta_B{}^*f'(0)$. Because gathering is not used at N_f, this reduces to $\theta_A{}^*f(N_f/\lambda)/(N_f/\lambda) \geq \theta_B{}^*f'(0)$. From (4.10) we have $\theta_A{}^*f(\underline{L}_A)/\underline{L}_A \equiv \theta_B{}^*f'(0)$ and together these imply $N_f/\lambda \leq \underline{L}_A$. However, by the construction of N_f, the definition of $\underline{k}$ in (4.10), and $k^0 < \underline{k}$ we have $k^0 = f'(N_f/\lambda)/g'(0) < f'(\underline{L}_A)/g'(0) = \underline{k}$. This implies $N_f/\lambda > \underline{L}_A$, which contradicts the previous result $N_f/\lambda \leq \underline{L}_A$. Therefore, $k^0 < \underline{k}$ implies $N_h < N_f$.

Now assume $k^0 > \underline{k}$ as in Proposition 4.7(b). We want to show that this implies $N_h > N_f$. Suppose instead $N_h \leq N_f$. By construction we have $L_B(N_h) = 0$ and $L_A(N_h) = N_h/\lambda$. Also, from the construction of N_h we have $\theta_A{}^*h(N_h/\lambda, k^0) = \theta_B{}^*f'(0)$. Because gathering is not used at N_h due to $N_h \leq N_f$, this reduces to $\theta_A{}^*f(N_h/\lambda)/(N_h/\lambda) = \theta_B{}^*f'(0)$. From (4.10) we have $\theta_A{}^*f(\underline{L}_A)/\underline{L}_A \equiv \theta_B{}^*f'(0)$ and thus $N_h/\lambda = \underline{L}_A$. However, from the construction of N_f, the definition of $\underline{k}$ in (4.10), and $k^0 > \underline{k}$ we have $k^0 = f'[L_A(N_f)]/g'(0) > f'(\underline{L}_A)/g'(0) = \underline{k}$. This implies that $L_A(N_f) < \underline{L}_A = N_h/\lambda = L_A(N_h)$. This is a contradiction because $N_h \leq N_f$ and $L_A(N)$ is increasing. Therefore, $k^0 > \underline{k}$ implies $N_h > N_f$.

Finally, assume $k^0 = \underline{k}$ as in Proposition 4.7(c). We want to show that this implies $N_h = N_f$. Suppose instead $N_h > N_f$. Using this strict inequality in the argument from two paragraphs above, we can show that $N_f/\lambda < \underline{L}_A$. Modifying the rest of the argument using $k^0 = \underline{k}$, we can show that $N_f/\lambda = \underline{L}_A$. This is a contradiction. Next, suppose instead $N_h < N_f$. Using the argument from one paragraph above, we can show that $L_A(N_h) = N_h/\lambda = \underline{L}_A$. Modifying the rest of the argument using $k^0 = \underline{k}$ gives $L_A(N_f) = \underline{L}_A = L_A(N_h)$. This is a contradiction because $N_h < N_f$ and $L_A(N)$ is an increasing function. Therefore, $k^0 = \underline{k}$ implies $N_h = N_f$.

(a) Consider Proposition 4.7(a). Assume $k^0 < \underline{k}$ and thus $N_h < N_f$. Let N^* be the LRE population for the new climate and the productivity k^0. There are three cases:

(a)(i) Suppose $N^* \leq N_h$. This implies $L_B{}^* = 0$ in LRE. $L_B{}^* = 0$ holds iff $\theta_B{}^*h(0, k^0) \leq y^*$. Because $h(0, k^0) \equiv \max \{f'(0), k^0g'(0)\} = f'(0)$, this reduces to $\theta_B{}^*f'(0) \leq y^*$.

(a)(ii) Suppose $N_h < N^* \leq N_f$. The inequality $N_h < N^*$ implies $L_B^* > 0$ in LRE. As above, this holds iff $\theta_B^* f'(0) > y^*$. The inequality $N^* \leq N_f$ implies $L_{Ag}^* = 0$, which holds iff $f'(L_A^*) \geq k^0 g'(0)$, where L_A^* is defined by the LRE condition $\theta_A^* h(L_A^*, k^0) \equiv y^*$ stated in Proposition 4.7.

(a)(iii) Suppose $N_f < N^*$. The fact that $N_h < N^*$ implies $L_B^* > 0$ in LRE. As above, this holds iff $\theta_B^* f'(0) > y^*$. $N_f < N^*$ implies $L_{Ag}^* > 0$, which holds iff $f'(L_A^*) < k^0 g'(0)$, where L_A^* is defined as in case (ii) above.

These cases are mutually exclusive and exhaustive, so the converses also hold:

(a)(i) If $\theta_B^* f'(0) \leq y^*$ then $N^* \leq N_h < N_f$.

(a)(ii) If $\theta_B^* f'(0) > y^*$ and $f'(L_A^*) \geq k^0 g'(0)$ then $N_h < N^* \leq N_f$.

(a)(iii) If $\theta_B^* f'(0) > y^*$ and $f'(L_A^*) < k^0 g'(0)$ then $N_h < N_f < N^*$.

The results in Proposition 4.7(a) are obtained as follows.

(a)(i) If $\theta_B^* f'(0) \leq y^*$ then $N^* \leq N_h < N_f$. We have $N^0 < N^*$ because climate amelioration implies that the baseline VLRE in Proposition 4.5 has a lower regional population than the new LRE. From A4.4 the regional population $\{N^t\}$ is increasing and $\{N^t\}$ approaches N^* in the limit. Thus, $N^t < N_h < N_f$ for all $t \geq 0$. It follows that type-B sites never become active and gathering is never used. The sedentism rate remains at zero because $L_B^t = 0$ for all $t \geq 0$.

(a)(ii) If $\theta_B^* f'(0) > y^*$ and $f'(L_A^*) \geq k^0 g'(0)$ then $N_h < N^* \leq N_f$. It must be true that $N^0 \leq N_h$ because type-B sites are not active in period $t = 0$ (by assumption the necessary condition for this to occur in Proposition 4.6(b) does not hold). Again, the regional population $\{N^t\}$ is increasing and $\{N^t\}$ approaches N^* in the limit. Thus, $N^t < N_f$ for all $t \geq 0$ and gathering is never used. However, there is some T

> 0 such that $N^t \leq N_h$ for t = 0, 1 . . T-1 and $N^t > N_h$ for t = T, T+1 . . Therefore, the type-B sites are not active for t < T but are active for t ≥ T. The sedentism rate has a positive limit $S^* = L_B^*/L_A^* < 1$ because $L_A(N)$ and $L_B(N)$ are continuous, N approaches N*, and $0 < L_B^* < L_A^*$.

(a)(iii) If $\theta_B^*f'(0) > y^*$ and $f'(L_A^*) < k^0g'(0)$ then $N_h < N_f < N^*$. Again, $N^0 \leq N_h$, the regional population $\{N^t\}$ is increasing, and $\{N^t\}$ approaches N* in the limit. As above there is some T > 0 such that $N^t \leq N_h$ for t = 0, 1 . . T-1 and $N^t > N_h$ for t = T, T+1 . . Therefore, the type-B sites are not active for t < T but are active for t ≥ T. In addition, there is some T′ ≥ T such that $N^t \leq N_f$ for t = 0, 1 . . T′-1 and $N^t > N_f$ for t = T′, T′+1 . . Therefore, gathering is not used for t < T′ but is used at sites of type A for t ≥ T′. The result for S* is obtained as in case (ii) above.

(b) Consider Proposition 4.7(b). Assume $k^0 > \underline{k}$ and thus $N_f < N_h$. Let N* be the LRE population for the new climate and the productivity k^0. There are three cases:

(b)(i) Suppose $N^* \leq N_f$. This implies $L_{Ag}^* = 0$ in LRE, which holds iff $f'(L_A^*) \geq k^0g'(0)$.

(b)(ii) Suppose $N_f < N^* \leq N_h$. The inequality $N_f < N^*$ implies $L_{Ag}^* > 0$ in LRE, which holds iff $f'(L_A^*) < k^0g'(0)$. The inequality $N^* \leq N_h$ implies $L_B^* = 0$ in LRE, which holds iff $\theta_B^*h(0, k^0) \leq y^*$. From $h(0, k^0) = f'(0)$ this reduces to $\theta_B^*f'(0) \leq y^*$.

(b)(iii) Suppose $N_h < N^*$. This implies $f'(L_A^*) < k^0g'(0)$ as above. $N_h < N^*$ implies $L_B^* > 0$ in LRE, which holds iff $\theta_B^*f'(0) > y^*$.

These cases are mutually exclusive and exhaustive, so the converses also hold:

(b)(i) If $f'(L_A^*) \geq k^0g'(0)$ then $N^* \leq N_f < N_h$.

(b)(ii) If $f'(L_A{*}) < k^0g'(0)$ and $\theta_B{*}f'(0) \leq y{*}$ then $N_f < N{*} \leq N_h$.

(b)(iii) If $f'(L_A{*}) < k^0g'(0)$ and $\theta_B{*}f'(0) > y{*}$ then $N_f < N_h < N{*}$.

The results in Proposition 4.7(b) are obtained as follows.

(b)(i) If $f'(L_A{*}) \geq k^0g'(0)$ then $N{*} \leq N_f < N_h$. We have $N^0 < N{*}$ because climate amelioration implies that the baseline VLRE in Proposition 4.5 has a lower regional population than the new LRE. From A4.4, regional population $\{N^t\}$ is increasing and $\{N^t\}$ approaches N* in the limit. Thus, $N^t < N_f < N_h$ for all $t \geq 0$. It follows that type-B sites are never active and gathering is never used. The sedentism rate remains at zero because $L_B^t = 0$ for all $t \geq 0$.

(b)(ii) If $f'(L_A{*}) < k^0g'(0)$ and $\theta_B{*}f'(0) \leq y{*}$ then $N_f < N{*} \leq N_h$. It must be true that $N^0 \leq N_f$ because gathering is not used in period t = 0 due to Proposition 4.6(a). Again, the regional population $\{N^t\}$ is increasing and $\{N^t\}$ approaches N* in the limit. Thus, $N^t < N_h$ for all $t \geq 0$ and type-B sites are never active. However, there is some T > 0 such that $N^t \leq N_f$ for t = 0, 1 . . T-1 and $N^t > N_f$ for t = T, T+1 . . Thus, gathering is not used for t < T but it is used at sites of type A for $t \geq T$. The sedentism rate remains at zero because $L_B^t = 0$ for all $t \geq 0$.

(b)(iii) If $f'(L_A{*}) < k^0g'(0)$ and $\theta_B{*}f'(0) > y{*}$ then $N_f < N_h < N{*}$. Again, $N^0 \leq N_f$, the regional population $\{N^t\}$ is increasing, and $\{N^t\}$ approaches N* in the limit. As above there is some T > 0 such that $N^t \leq N_f$ for t = 0, 1 . . T-1 and $N^t > N_f$ for t = T, T+1 . . Thus, gathering is not used for t < T but it is used at sites of type A for $t \geq T$. In addition, there is some $T' \geq T$ such that $N^t \leq N_h$ for t = 0, 1 . . T′-1 and $N^t > N_h$ for t = T′, T′+1 . . Thus, type-B sites are not active for t < T′ but are active for $t \geq T'$. The sedentism rate has a positive limit $S^* = L_B^*/L_A^* < 1$ because $L_A(N)$ and $L_B(N)$ are continuous, N approaches N*, and $0 < L_B^* < L_A^*$.

(c) Consider Proposition 4.7(c). Assume $k^0 = \underline{k}$ and thus $N_h = N_f$. Let N* be the LRE population for the new climate and the productivity k^0. There are two cases:

(c)(i) Suppose $N^* \leq N_f = N_h$. This implies $L_B^* = 0$ in LRE, which holds iff $\theta_B^*f'(0) \leq y^*$. It also implies that gathering does not occur at type-A sites in LRE, which is true iff $f'(L_A^*) \geq k^0g'(0)$.

(c)(ii) Suppose $N_f = N_h < N^*$. This implies $L_B^* > 0$ in LRE, which holds iff $\theta_B^*f'(0) > y^*$. It also implies that gathering does occur at type-A sites in LRE, which is true iff $f'(L_A^*) < k^0g'(0)$.

These cases are mutually exclusive and exhaustive, so the converses also hold:

(c)(i) If $\theta_B^*f'(0) \leq y^*$ and $f'(L_A^*) \geq k^0g'(0)$ then $N^* \leq N_f = N_h$.

(c)(ii) If $\theta_B^*f'(0) > y^*$ and $f'(L_A^*) < k^0g'(0)$ then $N_f = N_h < N^*$.

The results in Proposition 4.7(c) are obtained as follows.

(c)(i) If $\theta_B^*f'(0) \leq y^*$ and $f'(L_A^*) \geq k^0g'(0)$ then $N^* \leq N_f = N_h$. We have $N^0 < N^*$ due to climate amelioration. As in other cases, the regional population $\{N^t\}$ is increasing and approaches N* in the limit. Thus, $N^t < N_f = N_h$ for all $t \geq 0$. It follows that type-B sites are never active and gathering is never used. The sedentism rate remains at zero because $L_B^t = 0$ for all $t \geq 0$.

(c)(ii) If $\theta_B^*f'(0) > y^*$ and $f'(L_A^*) < k^0g'(0)$ then $N_f = N_h < N^*$. It must be true that $N^0 \leq N_f = N_h$ because gathering is not used in period $t = 0$ due to Proposition 4.6(a). The path $\{N^t\}$ has the same properties as in case (c)(i) above. Thus, there is some $T > 0$ such that $N^t \leq N_f = N_h$ for $t = 0, 1 \ldots T\text{-}1$ and $N^t > N_f = N_h$ for $t = T, T\text{+}1 \ldots$

It follows that type-B sites are not active and gathering is not used for $t < T$, but type-B sites become active and gathering is used at type-A sites for $t \geq T$. The sedentism rate has a positive limit $S^* = L_B^*/L_A^* < 1$ because $L_A(N)$ and $L_B(N)$ are continuous, N approaches N^*, and $0 < L_B^* < L_A^*$.

Proof of Proposition 4.8 (very long run; climate plus population and technology).

Because gathering never shuts down after it begins, the only possibility for VLRE involves gathering productivity at the level k*. The conditions for LRE require that sites of type B have $L_B^* = 0$ if $\theta_B^*h(0, k^*) \leq y^*$ and $L_B^* > 0$ if $\theta_B^*h(0, k^*) > y^*$.

In cases (a)(iii), (b)(iii), and c(ii) from Proposition 4.7, we have $\theta_B^*f'(0) > y^*$. By the definition in Proposition 4.1(f), $h(0, k^*) \equiv \max \{f'(0), k^*g'(0)\} \geq f'(0)$. Therefore, in all of these cases we have $\theta_B^*h(0, k^*) > y^*$. This implies $L_B^* > 0$ in the new VLRE.

In case (b)(ii) from Proposition 4.7, we have $\theta_B^*f'(0) \leq y^*$. From the definition of $h(0, k^*)$, the inequality $\theta_B^*h(0, k^*) > y^*$ holds iff $\theta_B^*k^*g'(0) > y^*$. This implies $L_B^* > 0$ iff the latter inequality holds.

Proof of Proposition 4.9 (persistence of mobile activities).

The baseline VLRE in Proposition 4.5 has $\theta_A^0 f'(0) > y^*$ due to (4.9a). Suppose only gathering is used at the type-A sites in the new VLRE. This implies $f'(0) \leq kg'(L_A^*)$ where L_A^* is the local population at type-A sites and k is the gathering productivity in the new VLRE. It is unimportant whether $k < k^*$ or $k = k^*$. If only gathering is used at the type-A sites, then LRE implies $\theta_A^* kg(L_A^*)/L_A^* = y^*$. Combining these results with the fact that the average product of gathering exceeds the marginal product, we obtain

$$y^* < \theta_A^0 f'(0) < \theta_A^* f'(0) \leq \theta_A^* kg'(L_A^*) < \theta_A^* kg(L_A^*)/L_A^* = y^*$$

This is a contradiction. It follows that type-A sites must use both hunting and gathering in the new VLRE.

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Chapter 5: The Transition to Agriculture

Proofs of Formal Propositions

Proof of Proposition 5.1 (optimal time allocation).

A solution $[n_f(n, c, s), n_g(n, c, s)]$ exists because the feasible set is non-empty and compact, and the objective function is continuous. The optimal allocation is unique due to convexity of the feasible set and strict concavity of the objective function. Continuity of the solution and of $H(n, c, s)$ follows from the theorem of the maximum.

To establish the strict concavity of $H(n, c, s)$ in n, fix $(c, s) > 0$. Choose any $n' \geq 0$ and $n'' \geq 0$ with $n' \neq n''$. Let (n_f', n_g') be optimal for n' and let (n_f'', n_g'') be optimal for n''. Choose some arbitrary $\mu \in (0, 1)$. Define $n_f^\mu \equiv \mu n_f' + (1-\mu)n_f''$ and $n_g^\mu \equiv \mu n_g' + (1-\mu)n_g''$. The allocation (n_f^μ, n_g^μ) is feasible when the population of the site is $n^\mu \equiv \mu n' + (1-\mu)n''$. This implies

$$\begin{aligned} H(n^\mu, c, s) &\geq F(n_f^\mu, c, s) + G(n_g^\mu, c, s) \\ &= F[\mu n_f' + (1-\mu)n_f'', c, s] + G[\mu n_g' + (1-\mu)n_g'', c, s] \\ &> \mu F(n_f', c, s) + (1-\mu)F(n_f'', c, s) + \mu G(n_g', c, s) + (1-\mu)G(n_g'', c, s) \\ &= \mu H(n', c, s) + (1-\mu)H(n'', c, s) \end{aligned}$$

where the weak inequality follows from the fact that H is a maximum; the first equality follows from the definitions of n_f^μ and n_g^μ; the strict inequality follows from the fact that F and G are both strictly concave and either $n_f' \neq n_f''$, $n_g' \neq n_g''$, or both due to $n' \neq n''$; and the second equality follows from the optimality of (n_f', n_g') for n' and the optimality of (n_f'', n_g'') for n''. This shows that $H(n, c, s)$ is strictly concave in n for fixed $(c, s) > 0$.

Finally, we want to show that $n_f(n, c, s)$ and $n_g(n, c, s)$ are differentiable. From the differentiability of F and G, this will show that $H(n, c, s)$ is also differentiable. By strict concavity, the first order conditions are necessary and sufficient for a solution. As discussed in the text in connection with A5.2, when $0 \leq n \leq n^a(c, s)$ with $n^a(c, s)$ defined as in (5.2), it is optimal to have $n_f = n$ and $n_g = 0$ in (5.1). If $n < n^a(c, s)$, we have $F_n(n, c, s) > G_n(0, c, s)$ so $n_f(n, c, s) = n$ and $n_g(n, c, s) = 0$, which is clearly differentiable. When $n > n^a(c, s)$, the first order conditions require $F_n(n_f, c, s) = G_n(n_g, c, s)$ and $n_f + n_g = n$ with $n_f > 0$ and $n_g > 0$. The implicit function theorem then establishes the differentiability of the solution in (5.1). It follows that the solution and H are differentiable except possibly at $n = n^a(c, s)$. Both are continuous at this boundary by the theorem of the maximum.

Proof of Lemma 5.1.

(a) Fix $(c, s) > 0$. From A5.2 and A5.3, there is a threshold $n^a > 0$ defined by $f_n(n^a) \equiv kg_n(0)$ that is independent of (c, s). As in the proof of Proposition 5.1, $0 \leq n \leq n^a$ implies that it is optimal to choose $n_f = n$ and $n_g = 0$. On the interval $0 < n \leq n^a$ we have $y(n, c, s) = A(c, s)f(n)/n$ where $A(c, s) > 0$ is a constant. From the strict concavity of $f(n)$ we have $f(n)/n > f'(n)$ for all $n > 0$. A5.2 and A5.3 imply $\lim_{n\to 0} f'(n) = +\infty$ so $\lim_{n\to 0} y(n, c, s) = +\infty$.

(b) For all $n > n^a$ the solution in (5.1) satisfies the first order conditions $f'(n_f) = kg'(n_g)$ with $n_f + n_g = n$ and $n_f > 0$, $n_g > 0$. The first order conditions imply that n_f and n_g are both increasing functions of n. If there is a finite upper bound for n_f as $n \to \infty$ then there must be a strictly positive lower bound for $f'(n_f)$. But then $n_g \to \infty$ as $n \to \infty$ implies $kg'(n_g) \to 0$ as $n \to \infty$, contradicting the first order conditions. A similar argument rules out an upper bound for n_g. Therefore, both $n_f \to \infty$ and $n_g \to \infty$ as $n \to \infty$.

The ratio $f(n)/n$ is decreasing from strict concavity. We want to show that $f(n)/n \to 0$ as $n \to \infty$. Suppose instead there is a positive lower bound $\varepsilon > 0$ such that $f(n)/n > \varepsilon$ for all $n \geq 0$. By the strict concavity, for any arbitrary $n_0 > 0$ we must have $f(n) \leq f(n_0) + f'(n_0)(n - n_0)$ for all $n \geq 0$. Therefore, we have $\varepsilon n < f(n) \leq f(n_0) + f'(n_0)(n - n_0)$ for all $n \geq 0$. But according to A5.2 and A5.3 we can choose a fixed n_0 large enough that $f'(n_0) < \varepsilon$. This gives $\varepsilon n > f(n_0) + f'(n_0)(n - n_0)$ if n is sufficiently large. The resulting contradiction shows that there is no lower bound $\varepsilon > 0$ and therefore $f(n)/n \to 0$ as $n \to \infty$. The same argument shows that $g(n)/n \to 0$ as $n \to \infty$. We have $y(n, c, s) = A(c, s)[f(n_f) + kg(n_g)]/n \leq A(c, s)[f(n) + kg(n)]/n$. Together these results imply $\lim_{n\to\infty} y(n, c, s) = 0$.

Proof of Proposition 5.2 (short-run equilibrium).

(a) We first show that $y(n, c, s) \equiv H(n, c, s)/n$ is decreasing in n for fixed $(c, s) > 0$. Recall from Proposition 5.1 that $H(n, c, s)$ is strictly concave in n for any fixed $(c, s) > 0$. Using $H(0, c, s) = 0$, strict concavity implies $H(\mu n, c, s) > \mu H(n, c, s)$ for all $n > 0$ and $\mu \in (0, 1)$. This gives $H(\mu n, c, s)/\mu n > H(n, c, s)/n$ for all $n > 0$ and $\mu \in (0, 1)$. Thus, $y(n, c, s)$ is decreasing in n. By Lemma 5.1, $\lim_{n\to 0} y(n, c, s) = +\infty$ and $\lim_{n\to\infty} y(n, c, s) = 0$.

These results imply that for fixed $(c, s) > 0$, $n(y, c, s)$ is decreasing in y with $\lim_{y\to 0} n(y, c, s) = +\infty$ and $\lim_{y\to\infty} n(y, c, s) = 0$. Drop the time subscripts in (5.4) and write the SRE condition as $N = \int_0^1 n(y, c, s)q(s)ds$ where $q(s) > 0$ for all $s \in [0, 1]$. The integral is a continuous and decreasing function of y, goes to infinity as $y \to 0$, and goes to zero as $y \to\infty$. It follows that for any $N > 0$ and $c > 0$, there is a unique $y = z(N, c)$ such that $N \equiv \int_0^1 n[z(N, c), c, s]q(s)ds$. It also follows that $z(N, c)$ is a decreasing function of N. The integral is increasing in c, so for a fixed $N > 0$, $z(N, c)$ is increasing in c.

(b) Again dropping the time subscripts, we need to show that $n[z(N, c), c, s]$ is an increasing function of s for any fixed $(N, c) > 0$. It is sufficient to show that $n(y, c, s)$ is increasing in s for fixed $(y, c) > 0$. By definition $y(n, c, s) \equiv H(n, c, s)/n$. For a fixed $n > 0$, the right-hand side is increasing in s because the functions $F(n_f, c, s)$ and $G(n_g, c, s)$ are increasing in s when all inputs are positive. Also, for a fixed $s > 0$, the right-hand side is decreasing in n due to previous results. Because we are holding y fixed, an increase in s implies an increase in n. Hence, $n(y, c, s)$ is increasing in s.

Proof of Proposition 5.3 (climate and migration in short-run equilibrium).

Let $c_0 > 0$ be the initial climate and let $c_1 > c_0$ be the new climate. The associated per capita food incomes are y_0 and y_1 where from the short run equilibrium condition (5.4) these are the unique solutions to $N = \int_0^1 n(y_0, c_0, s)q(s)ds = \int_0^1 n(y_1, c_1, s)q(s)ds$. We will show that $n[z(N, c_0), c_0, 1] > n[z(N, c_1), c_1, 1]$ or simply $n(y_0, c_0, 1) > n(y_1, c_1, 1)$.

From A5.3 we have $F(n_f, c, s) = A(c, s)f(n_f)$ and $G(n_g, c, s) = kA(c, s)g(n_g)$. This gives $y(n, c, s) = A(c, s)M(n)$ where $M(n) \equiv (1/n)\{\max f(n_f) + kg(n_g)$ subject to $n_f \geq 0$, $n_g \geq 0$, and $n_f + n_g = n\}$. Because $n(y, c, s)$ is the inverse of $y(n, c, s)$ for fixed $(c, s) > 0$, we have $y_0 \equiv A(c_0, 1)M[n(y_0, c_0, 1)]$ and $y_1 \equiv A(c_1,1)M[n(y_1, c_1, 1)]$. Because M is decreasing we have $n(y_0, c_0, 1) > n(y_1, c_1, 1)$ iff $M[n(y_0, c_0, 1)] < M[n(y_1, c_1, 1)]$, which in turn holds iff $y_0/A(c_0, 1) < y_1/A(c_1, 1)$.

Write $y_1 \equiv z(N, c_1) \equiv \theta z(N, c_0) \equiv \theta y_0$ where $\theta > 1$ because $c_1 > c_0$ and the function z is increasing in c for fixed $N > 0$. Suppose $A(c_1, s)/A(c_0, s)$ is decreasing in s (this will be proven below). We will show that this condition implies $A(c_1, 1) < \theta A(c_0, 1)$, which gives $y_0/A(c_0, 1) < y_1/A(c_1, 1)$ and hence the desired result $n(y_0, c_0, 1) > n(y_1, c_1, 1)$.

Suppose instead $A(c_1, s)/A(c_0, s)$ is decreasing in s but $A(c_1, 1) \geq \theta A(c_0, 1)$. This gives $A(c_1, s)/A(c_0, s) > \theta$ for all $s < 1$ and therefore $y_0/A(c_0, s) > y_1/A(c_1, s)$ for all $s < 1$. It follows that $y_0/A(c_0, s) = M[n(y_0, c_0, s)] > M[n(y_1, c_1, s)] = y_1/A(c_1 s)$ for all $s < 1$, which implies $n(y_0, c_0, s) < n(y_1, c_1, s)$ for all $s < 1$. But then the short run equilibrium condition (5.4) implies $N = \int_0^1 n(y_0, c_0, s)q(s)ds < \int_0^1 n(y_1, c_1, s)q(s)ds = N$. This contradiction shows that if $A(c_1, s)/A(c_0, s)$ is decreasing in s then $n(y_0, c_0, 1) > n(y_1, c_1, 1)$ as required.

Thus, it suffices to show that $A(c_1, s)/A(c_0, s)$ is decreasing in s. This is true if (i) $A(c_0, s)/A_s(c_0, s) < A(c_1, s)/A_s(c_1, s)$ for all s, where subscripts indicate differentiation. In turn, (i) is true if (ii) $A(c, s)/A_s(c, s)$ is increasing in c at each s. Differentiating with respect to c, (ii) is true if (iii) $A_c(c, s)A_s(c, s) > A(c, s)A_{cs}(c, s)$ for all (c, s). In the rest of the proof we show that (iii) holds if A(c, s) has constant returns to scale and an elasticity of substitution $\sigma > 1$, as assumed in A5.4.

Consider the expenditure minimization problem

$$\min p_s s + p_c c \quad \text{subject to } A(c, s) = A^0 \qquad \text{where } p_s > 0,\ p_c > 0.$$

Define $p \equiv p_s/p_c$. From the first order conditions we have the identities $pA_c[c(p), s(p)] \equiv A_s[c(p), s(p)]$ and $A[c(p), s(p)] \equiv A^0$. From constant returns, the marginal products A_c and A_s are homogeneous of degree zero. This gives $A_{cc} = -A_{cs}s/c$ and $A_{ss} = -A_{cs}c/s$. The second derivatives A_{cc} and A_{ss} are negative from A5.1, so the cross partial A_{cs} is positive.

The elasticity of substitution is $\sigma = -d[s(p)/c(p)]/dp \bullet p/[s(p)/c(p)] = -[(s'/s) - (c'/c)]p$. Differentiating the identities that define s(p) and c(p) yields $c'(p) = A_c/[2A_{cs} - (A_{ss}A_c/A_s) - (A_{cc}A_s/A_c)] > 0$ and $s'(p) = -A_c c'(p)/A_s < 0$. It follows that $\sigma > 0$.

Substituting these results into σ yields $\sigma > 1$ iff $A_c/s + A_s/c > 2A_{cs} - (A_{ss}A_c/A_s) - (A_{cc}A_s/A_c)$. Substituting $A_{cc} = -A_{cs}s/c$ and $A_{ss} = -A_{cs}c/s$ in this inequality shows that $\sigma > 1$ iff $A_cA_s > (cA_c + sA_s)A_{cs} = AA_{cs}$ where the equality follows from constant returns. This establishes condition (iii) in the preceding paragraph and completes the proof.

Proof of Proposition 5.4 (long-run equilibrium).

For the given $c > 0$, write $N^*(c) \equiv \int_0^1 n(y^*, c, s)q(s)ds$. From Lemma 5.1, the fact that y^* is positive and finite in A5.5, and $c > 0$, we have $n(y^*, c, s) > 0$ for all $s > 0$. Because $q(s) > 0$ for all s, we have $N^*(c) > 0$. Recall from Proposition 5.2(a) that there is a unique $y = z(N, c)$ such that (5.4) has $N \equiv \int_0^1 n(y, c, s)q(s)ds$. It follows that $y^* = z[N^*(c), c]$ is the unique SRE food per capita for the climate c and the population $N^*(c)$. Substituting this into D5.2 gives $\rho[z(N^*(c), c)] = \rho(y^*) = 1$, so $N^*(c) > 0$ is a LRE population for c. To show that this is the unique non-null LRE for c, suppose $N' \neq N^*(c)$ is another non-null LRE population for the same value of c. From D5.2 we have $\rho[z(N', c)] = 1$. This implies that $z(N', c) = y^* = z[N^*(c), c]$. But from Proposition 5.2(a), $z(N, c)$ is decreasing in N for a fixed $c > 0$, so these equalities contradict $N' \neq N^*(c)$. Hence, $N^*(c)$ is the unique non-null LRE population. $N^*(c)$ is increasing in c because y^* is a constant and $n(y^*, c, s)$ is increasing in c for all $s > 0$.

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Chapter 6: The Transition to Inequality

Proofs of Formal Propositions

Proof of Lemma 6.1.

Note that $0 < s^a(x) < s^b(x)$ for all $x > 0$.

(i) Suppose $L(s, x) < d$. Part (a) in D6.1 implies $L(s, x) = (s/x)^{1/(1-\alpha)} < d$ and thus $s < s^a(x)$.

(ii) Suppose $L(s, x) = d$. The first order condition for (b)(i) in D6.1 implies $\alpha s d^{\alpha-1} \leq x$ and thus $s \leq s^b(x)$. Part (b)(ii) in D6.1 implies $sd^\alpha - xd \geq 0$, and thus $s^a(x) \leq s$.

(iii) Suppose $L(s, x) > d$. The first order condition for (b)(i) in D6.1 implies $L(s, x) = (\alpha s/x)^{1/(1-\alpha)} > d$ and thus $s^b(x) < s$.

(iv) Suppose $s < s^a(x)$. If $L(s, x) = d$ then result (ii) implies $s^a(x) \leq s$, which is a contradiction. If $L(s, x) > d$ then result (iii) implies $s^a(x) < s^b(x) < s$, which is a contradiction. Thus result (i) applies and (a) in Lemma 6.1 is true.

(v) Suppose $s^a(x) \leq s \leq s^b(x)$. If $L(s, x) < d$ then result (i) implies $s < s^a(x)$, which is a contradiction. If $L(s, x) > d$ then result (iii) implies $s > s^b(x)$, which is a contradiction. Thus result (ii) applies and (b) in Lemma 6.1 is true.

(vi) Suppose $s^b(x) < s$. If $L(s, x) < d$ then result (i) implies $s < s^a(x) < s^b(x)$, which is a contradiction. If $L(s, x) = d$ then result (ii) implies $s \leq s^b(x)$ which is a contradiction. Thus result (iii) applies and (c) in Lemma 6.1 is true.

Proof of Lemma 6.2.

The "only if" statement follows from the definition of SRE in D6.1. Here we assume $N = \int_0^1 L(s, x)ds$ and prove the "if" statement. Suppose in what follows that conditions (a), (b), and (c) from Lemma 6.1 are all satisfied.

(a) If $L(s, x) < d$, Lemma 6.1(a) gives $L(s, x) = (s/x)^{1/(1-\alpha)}$. This gives $x = sL(s, x)^{\alpha-1}$. Thus part (a) in the definition of SRE from D6.1 is satisfied for the given x.

(b) If $L(s, x) = d$, Lemma 6.1(b) gives $s^a(x) \leq s \leq s^b(x)$. The latter inequality gives $\alpha s d^{\alpha-1} \leq x$, which is the first order condition for $L = d$ to be a solution in part (b)(i) of the definition of SRE from D6.1. This is sufficient for a maximum due to the concavity of the objective function. The former inequality gives $sd^\alpha - xd \geq 0$, so $r(s) \geq 0$ holds. Thus, (b)(ii) in the definition of SRE from D6.1 is also satisfied for the given x.

(c) If $L(s, x) > d$, Lemma 6.1(c) implies $L(s, x) = (\alpha s/x)^{1/(1-\alpha)}$ and $s^b(x) < s$. The latter inequality implies that L(s, x) obeys the first order condition to be a solution in part (b)(i) of the definition of SRE from D6.1. This is sufficient for a maximum due to the concavity of the objective function. Direct computation shows that $r(s) \geq 0$ also holds. Thus, (b)(ii) in the definition of SRE from D6.1 is also satisfied for the given x.

Proof of Proposition 6.1 (short-run equilibrium).

For parts (a), (b), and (c), we first prove the implications from x to D(x). The implications from N to x will be established at the end of the proof.

(a) $x^a < x$ implies $1 < s^a(x)$. All sites are open. The result for D(x) is obtained by integrating the density from Lemma 6.1(a) on [0, 1].

(b) $x^b \leq x \leq x^a$ implies $s^a(x) \leq 1 \leq s^b(x)$. All sites are either open, or closed but unstratified. The result for D(x) is obtained by integrating the density from Lemma 6.1(a) on $[0, s^a(x)]$ and the constant d from Lemma 6.1(b) on $[s^a(x), 1]$.

(c) $x < x^b$ implies $s^b(x) < 1$. Some sites are open, some are closed but unstratified, and some are stratified. The result for D(x) is obtained by integrating the density from Lemma 6.1(a) on $[0, s^a(x)]$, the constant d from Lemma 6.1(b) on $[s^a(x), s^b(x)]$, and the density from Lemma 6.1(c) on $[s^b(x), 1]$.

(d) Continuity of the derivative at x^a and x^b can be verified by direct computations. Continuity of the derivative elsewhere is obvious. It can also be shown through computations that the derivative is always negative. The limiting values of D(x) follow from the results in parts (a) and (c).

(e) The existence of a unique $x > 0$ such that $D(x) = N$ follows from $N > 0$, the limits of D(x), the continuity of D(x), and the fact that D(x) is decreasing. The fact that this x and the associated density $L(\cdot, x)$ from Lemma 6.1 form a SRE follows from Lemma 6.2. The implicit function theorem shows that the equilibrium wage x(N) is continuously differentiable with $x'(N) < 0$. The limit results for x(N) follow from the limit results for D(x) in part (d).

Using the result for D(x) in part (b), we have $N^a = D(x^a)$ and $N^b = D(x^b)$, or equivalently $x^a = x(N^a)$ and $x^b = x(N^b)$. Because x(N) is decreasing, $N < N^a$ implies $x^a < x(N)$; $N^a \leq N \leq N^b$ implies $x^b \leq x(N) \leq x^a$; and $N^b < N$ implies $x(N) < x^b$. This completes the proof for parts (a), (b), and (c) above.

Proof of Proposition 6.2 (aggregate production function).

(a) From Proposition 6.1, $N < N^a$ implies $x^a < x$. This gives $1 < s^a(x)$ so all sites are open. The result for $\phi(x)$ is obtained using the density $L(s, x) = (s/x)^{1/(1-\alpha)}$ from Lemma 6.1(a) on [0, 1].

(b) From Proposition 6.1, $N^a \leq N \leq N^b$ implies $x^b \leq x \leq x^a$. This gives $s^a(x) \leq 1 \leq s^b(x)$ so that all sites are either open, or closed but unstratified. The result for $\phi(x)$ is obtained using the density $L(s, x) = (s/x)^{1/(1-\alpha)}$ from Lemma 6.1(a) on $[0, s^a(x))$ and $L(s, x) = d$ from Lemma 6.1(b) on $[s^a(x), 1]$.

(c) From Proposition 6.1, $N^b < N$ implies $x < x^b$. This gives $s^b(x) < 1$ so that some sites are open, some sites are closed but unstratified, and some sites are stratified. The result for $\phi(x)$ is obtained using the density $L(s, x) = (s/x)^{1/(1-\alpha)}$ from Lemma 6.1(a) on $[0, s^a(x))$; $L(s, x) = d$ from Lemma 6.1(b) on $[s^a(x), s^b(x)]$; and $L(s, x) = (\alpha s/x)^{1/(1-\alpha)}$ from Lemma 6.1(c) on $(s^b(x), 1]$.

Continuity of $\phi'(x)$ at x^a and x^b can be verified by computation. Continuity of $\phi'(x)$ for all other $x > 0$ is obvious. It can be shown by computation that $\phi'(x) < 0$ for all $x > 0$. Part (e) of Proposition 6.1 ensures that $x'(N)$ is continuous and negative for all $N > 0$. Together these results imply that $Y'(N)$ is continuous and positive for all $N > 0$.

Proof of Corollary to Proposition 6.2.

(a) From Proposition 6.1(a) and $N = D(x)$ we have $x(N) = (Q/N)^{1/(1-\alpha)}$. Using the solution for $\phi(x)$ from Proposition 6.2(a) along with (6.3) gives the result.

(b) From Proposition 6.1(b) and $N = D(x)$ we have $x(N) = (d-N)(2-\alpha)d^{\alpha-2}$. Using the solution for $\phi(x)$ from Proposition 6.2(b) along with (6.3) gives the result.

(c) Consider $N > N^b$ so Proposition 6.2(c) applies. We have $Y'(N) = \phi'[x(N)]x'(N)$ and $Y''(N) = \phi''[x(N)][x'(N)]^2 + \phi'[x(N)]x''(N)$. From the identity $N \equiv D[x(N)]$,

$$x'(N) = 1/D'[x(N)] \quad \text{and} \quad x''(N) = -D''[x(N)]/\{D'[x(N)]\}^3.$$

Substituting these results into $Y''(N)$ gives

$$Y''(N) = -\{\phi'[x(N)]D''[x(N)] - \phi''[x(N)]D'[x(N)]\} / [D'(x(N))]^3.$$

Since $- 1/[D'(x(N))]^3 > 0$ from Proposition 6.1(d), the sign of $Y''(N)$ is the same as the sign of $\phi'[x(N)]D''[x(N)] - \phi''[x(N)]D'[x(N)]$. It therefore suffices to study the sign of $\phi'(x)D''(x) - \phi''(x)D'(x)$ on the interval $x < x^b$. Using Propositions 6.1(c) and 6.2(c), some algebra shows that this has the same sign as the quadratic $Av^2 + Bv + C$ where $v \equiv x^{1/Q}$, $A \equiv -d^{2(2-\alpha)}(1+\alpha)/\alpha^2$, $B \equiv d^{2-\alpha}\alpha^{\alpha/(1-\alpha)}(1-\alpha)^{-2}(2+\alpha-2\alpha^2)$, and $C \equiv -\alpha^{2/(1-\alpha)}(1-\alpha)^{-2}$. The quadratic is negative at $v = 0$ and positive at $v^b \equiv (x^b)^{1/Q}$. Since the quadratic is either rising throughout $[0, v^b)$ or has an interior maximum on this interval, there is a unique $v^c \in (0, v^b)$ at which the quadratic is zero, with a negative sign for all $v \in (0, v^c)$ and a positive sign for all $v \in (v^c, v^b)$. Thus, there is a unique $x^c = (v^c)^Q \in (0, x^b)$ such that $\phi'(x)D''(x) - \phi''(x)D'(x) < 0$ for $x \in (0, x^c)$, $= 0$ for $x = x^c$, and > 0 for $x \in (x^c, x^b)$. Finally, this implies that there is a unique $N^c = N(x^c) > N^b$ such that $Y''(N) > 0$ for $N \in (N^b, N^c)$, $= 0$ for $N = N^c$, and < 0 for $N > N^c$.

(d) Continuous differentiability of Y(N)/N follows from continuous differentiability of Y(N), which was established in Proposition 6.2. We want to show that Y(N)/N is globally decreasing. This can be done by direct computation for $0 < N \leq N^b$ using the results in parts (a) and (b) of the Corollary. Suppose $N > N^b$, which implies $x < x^b$ and $v < v^b$ in the notation used in the proof of (c) above. We will show that $Y'(N) - Y(N)/N < 0$ for all $N > N^b$. Using (6.3), the SRE identity $N \equiv D[x(N)]$, and the implicit function theorem, we obtain $Y'(N) - Y(N)/N = \phi'[x(N)]/D'[x(N)] - \phi[x(N)]/D[x(N)]$. Because $D'[x(N)] < 0$ from Proposition 6.1, this expression is opposite in sign to $\phi'[x(N)]D[x(N)] - \phi[x(N)]D'[x(N)]$. Thus, it suffices to show that the latter expression is positive for all $N > N^b$, or equivalently that $\phi'(x)D(x) - \phi(x)D'(x) > 0$ for all $x < x^b$. Some algebra shows that this is true iff $av^2 + bv + c > 0$ for all $v < v^b$ where $v \equiv x^{1/Q}$, $a \equiv d^{2(2-\alpha)}(1+\alpha)/2\alpha^{1/Q}$, $b \equiv d^{2-\alpha}(1/2 + \alpha - 1/\alpha)$, and $c \equiv \alpha^{1/(1-\alpha)}$. Since $a > 0$, this quadratic has a minimum value at $v_{min} = -b/2a$. There are three cases: (i) $v_{min} \leq 0$; (ii) $0 < v_{min} \leq v^b$; and (iii) $v^b < v_{min}$. In case (i), the quadratic is equal to $c > 0$ at $v = 0$ and positive for all $v > 0$. This yields the result. In case (ii), it suffices to show that the value of the quadratic is positive at v_{min}. This is true from $v_{min} \leq v^b$. In case (iii), the quadratic is positive at v^b, which implies that it is positive on $[0, v^b]$. This establishes that Y(N)/N is decreasing as claimed. The result $\lim_{N\to 0} Y(N)/N = \infty$ follows from part (a) of the Corollary. To show that $\lim_{N\to\infty} Y(N)/N = 0$, note that $Y(N)/N = \phi[x(N)]/D[x(N)]$ where $N \to \infty$ implies $x \to 0$ from Proposition 6.1. Computing this ratio as a function of x using Proposition 6.1(c) for the denominator and Proposition 6.2(c) for the numerator gives the desired result.

Proof of Proposition 6.3 (long-run equilibrium).

(a) The SRE conditions in D6.1 are built into the definition of the aggregate production function $Y(N; \theta)$ from (6.2). It suffices to show that for a given productivity $\theta > 0$, there is a unique $N(\theta) > 0$ such that $Y[N(\theta); \theta]/N(\theta) = 1/\rho$. This follows from the results in part (d) of the Corollary to Proposition 6.2.

(b) The LRE condition in (a) above and the implicit function theorem imply $N'(\theta) = Y_\theta[N(\theta), \theta] / \{Y[N(\theta); \theta]/N(\theta) - Y_N[N(\theta); \theta]\} > 0$ where the subscripts indicate partial derivatives. The inequality holds because the numerator is positive due to Proposition 6.2, and the denominator is positive due to part (d) of the Corollary. Continuity of $N'(\theta)$ follows from results in Proposition 6.2 and the Corollary. For the limits, note that (6.3) in the text and the definition of LRE together imply $1/\theta\rho = \int_0^1 sL(s, x)^\alpha \, ds / D(x)$. Consider $\theta \to 0$, which implies that the left-hand side $\to \infty$. Using Lemma 6.1 and Proposition 6.1, the right-hand side $\to \infty$ iff $x \to \infty$. Thus, $\theta \to 0$ implies $x \to \infty$. From Proposition 6.1(e), this implies $N \to 0$. This establishes $\lim_{\theta\to 0} N(\theta) = 0$. The other limit result is obtained through similar reasoning.

(c) From Proposition 6.1, N^a and N^b are positive constants that do not depend on θ. Proposition 6.3(b) implies that there are unique productivity levels such that $N(\theta^a) = N^a$ and $N(\theta^b) = N^b$, with $0 < \theta^a < \theta^b$ because $0 < N^a < N^b$. The first sentences of (i), (ii), and (iii) are immediate from the fact that $N(\theta)$ is increasing. The second sentence of (i) results from the fact that $N < N^a$ implies $x^a < x$ due to Proposition 6.1, and thus $1 < s^a(x)$ in Lemma 6.1(a). The second and third sentences of (ii) result from the fact that $N^a \leq N \leq N^b$ implies $x^b \leq x \leq x^a$ due to Proposition 6.1, and thus $0 < s^a(x) \le 1 \le s^b(x)$ in Lemma 6.1(b). The second and third sentences of (iii) result from the fact that $N^b < N$ implies $x < x^b$ due to Proposition 6.1, and thus $0 < s^a(x) < s^b(x) < 1$ in Lemma 6.1(c). With minor notational abuse, define $s^a(\theta) \equiv s^a[x(N(\theta))]$ and $s^b(\theta) \equiv s^b[x(N(\theta))]$ as in Lemma 6.1. These functions are continuously differentiable because $N(\theta)$ and $x(N)$ are continuously differentiable. They are decreasing because $N(\theta)$ is increasing and $x(N)$ is decreasing. The limit results follow from the limit results in (b) above, the limit results in Proposition 6.1(e), and the definitions of $s^a(x)$ and $s^b(x)$.

(d) Continuous differentiability of $w(\theta) = \theta x[N(\theta)]$ follows from the continuous differentiability of $x(N)$ and $N(\theta)$. When $\theta \le \theta^a$ we have $N \le N^a$, $x^a \le x$, and $1 \le s^a$. All agents at all sites receive the food income w (including at s = 1 if $1 = s^a$ because the marginal site has zero rent). Thus, LRE implies $w = Y(N; \theta)/N = 1/\rho$. For the rest of the proof we assume $\theta^a < \theta$ so that $N^a < N(\theta)$. We need to show that $w'(\theta) = x[N(\theta)] + \theta x'[N(\theta)]N'(\theta) < 0$. Upon substituting the result from part (b) above for $N'(\theta)$, differentiating (6.3) to obtain the marginal product $Y_N(N; \theta)$, using the linearity of $Y(N; \theta)$ as a function of θ in Proposition 6.2 to eliminate the partial derivative $Y_\theta(N; \theta)$, and using the implicit function theorem to obtain $x'(N) = 1/D'[x(N)]$ from Proposition 6.1(e), a necessary and sufficient condition for the desired result is $\phi(x)[xD'(x) + D(x)] > x\phi'(x)D(x)$ for all relevant values of x. In the case where $\theta^a < \theta \le \theta^b$ we have $N^a < N \le N^b$ and $x^b \le x < x^a$. Differentiating the functions from Propositions 6.1(b) and 6.2(b) yields an inequality involving a quadratic in x, which is satisfied for $x^b \le x < x^a$. The other case is $\theta^b < \theta$, where we have $N^b < N$ and $x < x^b$. Differentiating the functions from Propositions 6.1(c) and 6.2(c) yields an inequality that does not involve x and holds whenever $\alpha < 1$.

Proof of Proposition 6.4 (inequality).

Using (6.4), we define z^a and z^b so that $s^a = s(z^a)$ and $s^b = s(z^b)$.

(a) From (6.7), $y_1(z) > y_2(z)$ for all $0 < z < 1$ implies $G_1 < G_2$, so it suffices to establish the first claim. We begin by considering the derivative $y'(z)$ of the Lorenz curve $y(z)$ in (6.6). The fraction of agents in the regional population who have the lowest income w is $z^a \equiv (D_O + D_C)/N$ where D_O is the number of agents at open sites and D_C is the number of commoner agents at stratified sites. The fraction of regional income going to this set of agents is $y^a \equiv w(D_O + D_C)/Y$. Thus, for $z \in [0, z^a]$ we have $y'(z) = y^a/z^a = wN/Y$ and the Lorenz curve is linear.

For $z \in [z^a, 1]$ the derivative is $y'(z) = (N/Y)[w + r(s(z))/d]$ from (6.4), (6.5), and (6.6). The first derivative is continuous at z^a because $s(z^a) = s^a$ and $r(s^a) = 0$. For $z \in [z^a, z^b]$ the optimal labor input is $L(s) = d$ by Lemma 6.1(b), yielding $y'(z) = (\theta N/Y)d^{\alpha-1}s(z)$. From (6.4), this is linear and increasing in z. For $(z^a, z^b]$ we have $y''(z) = (\theta N^2/Y)d^{\alpha-2} > 0$ which is independent of z so that $y(z)$ is quadratic on this interval. The second derivative $y''(z)$ is discontinuous at z^a, where it jumps from zero to a positive number.

Whenever the interval $(z^b, 1]$ is non-empty, the envelope theorem can be used to disregard effects operating through the optimal labor input $L(s)$, and this yields $y''(z) = (\theta N^2/Yd^2)L[s(z)]^\alpha > 0$. This is larger than the second derivative on the quadratic interval $(z^a, z^b]$ because $L[s(z)] > d$ from Lemma 6.1(c), and it is increasing in z because $s(z)$ and $L(s)$ are both increasing. The first derivative $y'(z)$ is continuous at z^b because $s(z)$ and $r(s)$ are both continuous. Likewise, $y''(z)$ is continuous at z^b because $L(s)$ is continuous.

To compare the two Lorenz curves $y_1(z)$ and $y_2(z)$ from Proposition 6.4, we need to know how $y'(z)$ and $y''(z)$ respond to changes in N. We first show that the linear part of the Lorenz curve becomes flatter when N increases. For $N \in (N^a, N^b]$ the ratio wN/Y can be expressed in terms of x using Propositions 6.1(b) and 6.2(b). Differentiating with respect to x shows that wN/Y is increasing in x when a certain quadratic expression involving x is positive. This requirement is satisfied whenever $x^b \leq x < x^a$, which follows from $N \in (N^a, N^b]$. Since wN/Y is increasing in x, it is decreasing in N. For $N > N^b$, the ratio wN/Y can be expressed in terms of x using Propositions 6.1(c) and 6.2(c). Algebra and differentiation show that the ratio is increasing in x whenever $\alpha < 1$. Thus, the ratio is again decreasing in N. These results prove that $y_1'(z) > y_2'(z)$ whenever both curves are linear; that is, for the non-degenerate interval $0 \leq z \leq \min\{z_1^a, z_2^a\}$.

Now consider the slope of the Lorenz curve at z = 1: that is, $y'(1)$. When $N \in (N^a, N^b]$ we have $L(1) = d$ and $y'(1) = (\theta N/Y)d^{\alpha-1}$. The productivity parameter θ cancels with θ in the output expression $\phi(x)$ from Proposition 6.2(b), so this parameter does not affect the slope $y'(1)$. Part (d) of the Corollary from Section 6.6 shows that Y/N is decreasing in N. Therefore, the ratio in parentheses is increasing in N, and $y'(1)$ is increasing in N. When $N > N^b$, using $L(1) = (\alpha/x)^{1/(1-\alpha)}$ from Lemma 6.1(c) gives $y'(1) = (\theta N/Yd)[xd + (1-\alpha)(\alpha/x)^{\alpha/(1-\alpha)}]$. As before, $\theta N/Y$ is increasing in N and the value of θ does not affect $y'(1)$. The expression in brackets is decreasing in x whenever $x < x^b$, which follows from $N > N^b$. Therefore, the expression in brackets is increasing in N, and $y'(1)$ is increasing in N. We note that $y'(1)$ is continuous with respect to N at N^b. These results show that $y_1'(1) < y_2'(1)$, and by continuity that $y_1'(z) < y_2'(z)$ for all z in a non-degenerate neighborhood of z = 1.

By continuity, $y_1(z)$ - $y_2(z)$ has a maximum value at some $z^* \in [0, 1]$. The results for $y_1'(z)$ and $y_2'(z)$ in the last two paragraphs together with $y_1(0) = y_2(0) = 0$ and $y_1(1) = y_2(1) = 1$ show that $y_1(z) - y_2(z)$ is strictly positive on a neighborhood of $z = 0$ and also on a neighborhood of $z = 1$. Thus, for any maximizer z^* we have $y_1(z^*) - y_2(z^*) > 0$ where z^* is interior. Accordingly, there must be at least one local maximizer of $y_1(z) - y_2(z)$ on $z \in (0, 1)$ at which $y_1'(z^*) - y_2'(z^*) = 0$.

Suppose there is some interior point $z^0 \in (0, 1)$ at which $y_1(z^0) - y_2(z^0) = 0$ so the Lorenz curves intersect. This implies that there are at least two distinct interior local maxima separated by an interior local minimum. We will show that this is impossible.

First consider $N^a < N_1 < N_2 \leq N^b$. From previous results $y_1(z)$ is linear on $[0, z_1^a]$ and quadratic on $(z_1^a, 1]$. Likewise, $y_2(z)$ is linear on $[0, z_2^a]$ and quadratic on $(z_2^a, 1]$. It can be shown that z^a is a decreasing function of N on the interval $[N^a, N^b]$, which implies $0 < z_2^a < z_1^a$. Therefore, $y_1'(z) - y_2'(z)$ is a positive constant on $[0, z_2^a]$, and it decreases on $(z_2^a, z_1^a]$ because y_2 becomes quadratic while y_1 remains linear. For $(z_1^a, 1]$, both y_1 and y_2 are quadratic. Previous results give $y'' = \theta N^2/Yd^{\alpha-2}$ whenever a Lorenz curve is quadratic. We have shown that the ratio $\theta N/Y$ does not depend on θ and is increasing in N. Thus, y'' is increasing in N and $y_1''(z) < y_2''(z)$ on $(z_1^a, 1]$. This implies $y_1'(z) - y_2'(z)$ is decreasing on $(z_1^a, 1]$. Because $y_1'(z) - y_2'(z)$ is initially a positive constant and decreases thereafter, there cannot be more than one $z \in (0, 1)$ at which $y_1'(z) - y_2'(z) = 0$, so the Lorenz curves cannot intersect at an interior point. This proves part (a) for the case $N^a < N_1 < N_2 \leq N^b$.

Next, suppose $N^a < N_1 \leq N^b < N_2$. As in the preceding paragraph, $y_1(z)$ is linear on $[0, z_1^a]$ and quadratic on $(z_1^a, 1]$. Ignoring possible equalities among the boundaries z_1^a, z_2^a, and z_2^b, which do not affect the argument, there are three cases:

(i) $0 < z_1^a < z_2^a < z_2^b < 1$

(ii) $0 < z_2^a < z_1^a < z_2^b < 1$

(iii) $0 < z_2^a < z_2^b < z_1^a < 1$

For case (i), $y_1'(z) - y_2'(z)$ is initially a positive constant; then increases because y_1 becomes quadratic while y_2 remains linear; then decreases because both are quadratic (repeating a previous argument); and continues to decrease because y_1 remains quadratic while the second derivative of y_2 increases beyond quadratic. Thus, there cannot be more than one $z \in (0, 1)$ at which $y_1'(z) - y_2'(z) = 0$.

For case (ii), $y_1'(z) - y_2'(z)$ is initially a positive constant; then decreases because y_1 remains linear while y_2 becomes quadratic; then decreases because both are quadratic; and continues to decrease because y_1 remains quadratic while the second derivative of y_2 increases beyond quadratic. Thus, there cannot be more than one $z \in (0, 1)$ at which $y_1'(z) - y_2'(z) = 0$.

For case (iii), $y_1'(z) - y_2'(z)$ is initially a positive constant; then decreases because y_1 remains linear while y_2 becomes quadratic; then decreases because y_1 remains linear while the second derivative of y_2 increases beyond quadratic; and then decreases because y_1 is quadratic while y_2 is beyond quadratic (note in the last case that $y_1'(z) - y_2'(z)$ would be decreasing if both functions were quadratic, so this must also be true when the second derivative of y_2 is even larger). Thus, there cannot be more than one $z \in (0, 1)$ at which $y_1'(z) - y_2'(z) = 0$. We have therefore shown that the Lorenz curves cannot intersect at an interior point in any of cases (i), (ii), or (iii) when $N^a < N_1 \leq N^b < N_2$. This completes the proof of part (a) in Proposition 6.4.

(b) To compute the Gini coefficient in (6.7), we need to compute $\int_0^1 y(z)\, dz$. From (6.6), this has two components, one involving the wage and the other involving land rent. The integral involving the wage is wN/2Y. Using Propositions 6.1(b) and 6.2(b) with $N^b = 2Qd$ and $x^b = \alpha d^{\alpha-1}$ gives $wN/2Y = 2\alpha / (2+\alpha+\alpha^2)$ when this integral is evaluated at N^b. Using Propositions 6.1(c) and 6.2(c), wN/2Y approaches $\alpha/2$ as $N \to \infty$ and $x \to 0$.

To compute the rent component of $\int_0^1 y(z)\, dz$ in the insider-outsider range where $N^a < N \leq N^b$ and $x^b \leq x < x^a$, we calculate $R(z) = \int_0^{s(z)} r(s)\, ds$ for $z^a \leq z \leq 1$ as in (6.5). Because no sites are stratified, $r(s) = 0$ for $0 \leq s \leq s^a$ and $r(s) = \theta(sd^\alpha - xd)$ for $s^a \leq s \leq 1$. We then compute $\int_0^1 R(z)\, dz$ where $R(z) = 0$ for $0 \leq z \leq z^a$ and $R(z) = \int_0^{s(z)} r(s)\, ds$ for $z^a \leq z \leq 1$. Dividing by $\phi(x)$ from Proposition 6.2(b) gives $\int_0^1 R(z)\, dz / Y$. By $N^b = 2Qd$ and $x^b = \alpha d^{\alpha-1}$, at N^b this ratio is $\int_0^1 R(z)\, dz / Y = (2-\alpha)^2[(1-\alpha^3)/3(1-\alpha) - \alpha] / 2(1-\alpha)(2+\alpha+\alpha^2)$. Adding this to the result in the preceding paragraph for wN/2Y to get $\int_0^1 y(z)\, dz$ at N^b and then using (6.7) to compute the Gini coefficient yields the result for $G(N^b)$ stated in Proposition 6.4(b). This is the upper bound for insider-outsider inequality because part (a) showed that the Gini is increasing in N on the interval $(N^a, N^b]$. It is the lower bound for elite-commoner inequality because (a) showed that we have Lorenz curve dominance between N^b and any $N_2 > N^b$.

To compute the rent component of $\int_0^1 y(z)\, dz$ in the elite-commoner range where $N^b < N$ and $x < x^b$, we calculate $R(z) = \int_0^{s(z)} r(s)\, ds$ for $z^a \leq z \leq 1$ as in (6.5). Because some sites are now stratified, $r(s) = 0$ for $0 \leq s \leq s^a$; $r(s) = \theta(sd^\alpha - xd)$ for $s^a \leq s \leq s^b$; and $r(s) = \theta[s(\alpha s/x)^{\alpha/(1-\alpha)} - x(\alpha s/x)^{1/(1-\alpha)}]$ for $s^b < s \leq 1$. We then compute $\int_0^1 R(z)\, dz$ where

$R(z) = 0$ for $0 \leq z \leq z^a$ and $R(z) = \int_0^{s(z)} r(s)\, ds$ for $z^a \leq z \leq 1$. Dividing by $\phi(x)$ from Proposition 6.2(c) gives $\int_0^1 R[s(z)]\, dz / Y$. Letting $N \to \infty$ and $x \to 0$, it can be shown that this ratio approaches zero. Combining this with the earlier result $wN/2Y \to \alpha/2$ implies $\int_0^1 y(z)\, dz \to \alpha/2$. Using (6.7) to compute the Gini coefficient yields the limit result stated in Proposition 6.4(b).

Proof of Proposition 6.5 (demography).

Landless agents do not replace themselves because $w < 1/\rho$ in Proposition 6.3(d). The inequality $\theta^a < \theta$ gives $N > N^a$ from Proposition 6.3 and $x < x^a$ from Proposition 6.1. The latter result gives $s^a < 1$. The inequality $s^a < s^r$ follows from $w < 1/\rho$ and $x = w/\theta$.

Suppose $\theta^a < \theta \leq \theta^b$ so that $N^a < N \leq N^b$, $x^b \leq x < x^a$, and $s^a < 1 \leq s^b$ (no sites are stratified). We want to show that $s^r < 1$. This is true if $1/\rho = Y(N)/N < \theta d^{\alpha-1}$ where the equality follows from the definition of LRE in D6.2. The inequality can be expressed in terms of x as $\phi(x)/D(x) < \theta d^{\alpha-1}$ where the ratio on the left is obtained from Propositions 6.1(b) and 6.2(b). This reduces to a quadratic expression in x that is positive. It can be shown that the latter expression is decreasing on $[x^b, x^a)$ and zero at x^a. This gives $s^r < 1$.

Suppose instead that $\theta^b < \theta$ so that $N^b < N$, $x < x^b$, and $s^a < s^b < 1$ (some sites are stratified). We want to show that $s^r < s^b$. This is true if $1/\rho = Y(N)/N < \theta x/\alpha$ where the equality follows from the definition of LRE in D6.2. The inequality can be expressed in terms of x as $\phi(x)/D(x) < \theta x/\alpha$ where the ratio on the left is obtained from Propositions 6.1(c) and 6.2(c). Some algebra shows that this is true when $\alpha < 1$. This gives $s^r < s^b$.

The result $s^r \in (s^a, s^b)$ implies $L(s^r) = d$ from Lemma 6.1. From the definition of SRE in D6.1 and the definition of s^r in Proposition 6.5, each insider at s^r has the income $w + r(s^r)/d = 1/\rho$. Hence, these insiders exactly replace themselves. The remainder of Proposition 6.5 follows from the fact that r(s) is continuous and increasing.

Economic Prehistory: Six Transitions That Shaped The World

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Chapter 7: Warfare Between Egalitarian Groups

Proofs of Formal Propositions

Proof of Lemma 7.1.

(a) From (7.8) $x_A(\sigma) + x_B(\sigma) = (s_A + s_B)/\phi(s_A, s_B)$, where $\phi(s_A, s_B) \equiv s_A/[1 + \sigma^{-1/(1-\alpha)}]^\alpha + s_B/[1 + \sigma^{1/(1-\alpha)}]^\alpha$ from (7.5b). Using $\sigma \equiv s_A/s_B > 0$ from (7.5a), some algebra shows that $x_A(\sigma) + x_B(\sigma) > 1$ iff the following expression is positive:

$$\sigma[(2 + \sigma^{-1/(1-\alpha)} + \sigma^{1/(1-\alpha)})^\alpha - (1 + \sigma^{1/(1-\alpha)})^\alpha]$$

$$+ (2 + \sigma^{-1/(1-\alpha)} + \sigma^{1/(1-\alpha)})^\alpha - (1 + \sigma^{-1/(1-\alpha)})^\alpha$$

This is true because each line is strictly positive.

(b) From (7.8) we have $x_A(\sigma) \equiv s_A/\phi(s_A, s_B)$. From (7.4) and (7.5c) this can be rewritten as

$x_A(\sigma) = s_A / \max \{s_A L_A^\alpha + s_B L_B^\alpha \text{ subject to } L_A \geq 0, L_B \geq 0, L_A + L_B = 1\}$ or

$x_A(\sigma) = 1 / \max \{L_A^\alpha + (1/\sigma) L_B^\alpha \text{ subject to } L_A \geq 0, L_B \geq 0, L_A + L_B = 1\}$

The right-hand side is increasing in σ due to the envelope theorem. Thus $x_A'(\sigma) > 0$ for all $\sigma > 0$. A similar argument shows that $x_B'(\sigma) < 0$ for all $\sigma > 0$.

(c) Write $x_A(\sigma) = 1 / \max \{L_A^\alpha + (1/\sigma) L_B^\alpha \text{ subject to } L_A \geq 0, L_B \geq 0, L_A + L_B = 1\}$ as in (b). Because multiplication of the objective function by the constant $(1/s_A)$ has no effect on the solution, the optimal L_A and L_B in the denominator are given by (7.5d) for $n = 1$. Making this substitution, it can be shown that the denominator of $x_A(\sigma)$ approaches infinity as $\sigma \to 0$ and approaches 1 as $\sigma \to \infty$. This gives $x_A(0) = 0$ and $x_A(\infty) = 1$.

(d) The proof parallels (c).

Proof of Proposition 7.1 (war and peace).

Attack is a dominant strategy for A iff $p_A h(n_A) > s_A n_A^{\alpha-1}$ or equivalently $n_A/N > x_A(\sigma)$. Attack is a dominant strategy for B iff $p_B h(n_B) > s_B n_B^{\alpha-1}$ or equivalently $n_B/N > x_B(\sigma)$. When neither inequality holds, there is peace as in (7.7). If both A and B attack, then $n_A/N + n_B/N > x_A(\sigma) + x_B(\sigma) > 1$ where the second inequality is obtained from (7.9a). This is impossible because $n_A + n_B = N$. Thus A and B cannot both attack. The results in (a), (b), and (c) follow from the first three sentences above.

To show that equality of marginal products implies peace, fix $N > 0$ and $\sigma > 0$. Write $L_A^* = N/[1 + \sigma^{-1/(1-\alpha)}]$ and $L_B^* = N/[1 + \sigma^{1/(1-\alpha)}]$ as in (7.5d), where $L_A^* + L_B^* = N$. This is the unique labor allocation that equates marginal products across sites, and it is also the unique allocation that equates average products across sites. Let the total food output from (L_A^*, L_B^*) be

$$Y^* = s_A(L_A^*)^{\alpha} + s_B(L_B^*)^{\alpha}$$

$$= H(N) = \max \{s_A L_A^{\alpha} + s_B L_B^{\alpha} \text{ subject to } L_A \geq 0, L_B \geq 0, L_A + L_B = N\}.$$

Because the average products are equal, we have $Y^*/N = s_A(L_A^*)^{\alpha-1} = s_B(L_B^*)^{\alpha-1}$. Peace is strictly better than war for group A when $Y^*/N = s_A(L_A^*)^{\alpha-1} > (L_A^*/N)h(L_A^*) = H(L_A^*)/N$. This holds because $L_A^* < N$ gives $H(L_A^*) < H(N) = Y^*$. The proof is similar for B. This shows that peace is strictly better for each group, so case (b) holds with strict inequalities.

Proof of Proposition 7.2 (interior locational equilibria).

For any given allocation n there are three possibilities: (a) B attacks; (b) there is peace; or (c) A attacks. Using Proposition 7.1, we consider each case in turn.

(a) B attacks iff $n_A/N < 1\text{-}x_B$ or equivalently $n_A/n_B < (1\text{-}x_B)/x_B$. The utility functions are those from the warfare case in (7.10). The inequalities in (7.11) yield $\eta^{1/\alpha} \leq n_A/n_B \leq (1/\eta)^{1/\alpha}$. Together this gives Proposition 7.2(a).

(b) There is peace iff $1\text{-}x_B \leq n_A/N \leq x_A$ or equivalently $(1\text{-}x_B)/x_B \leq n_A/n_B \leq x_A/(1\text{-}x_A)$. The utility functions are those from the peace case in (7.10). The inequalities in (7.11) yield $(\sigma\eta)^{1/(1-\alpha)} \leq n_A/n_B \leq (\sigma/\eta)^{1/(1-\alpha)}$. Together this gives Proposition 7.2(b).

(c) A attacks iff $x_A < n_A/N$ or equivalently $x_A/(1\text{-}x_A) < n_A/n_B$. The utility functions are those from the warfare case in (7.10). The inequalities in (7.11) yield $\eta^{1/\alpha} \leq n_A/n_B \leq (1/\eta)^{1/\alpha}$. Together this gives Proposition 7.2(c).

Proof of Proposition 7.3 (migration).

(a) Follows from D7.1, the construction of LE_B, LE_P, and LE_A in Proposition 7.2, and D7.2(a).

(b) Suppose LE_B is empty and consider two possibilities:

(i) $m_A/m_B < (1-x_B)/x_B$, which yields war; and

(ii) $(1-x_B)/x_B \leq m_A/m_B$ with m_A/m_B below LE_P, which yields peace.

In case (i), the utility functions for war in (7.10) give $u_A(n) = [\phi(s_A, s_B)/N]n_A^\alpha$ and $u_B(n) = [\phi(s_A, s_B)/N]n_B^\alpha$ for all n with $n_A \leq m_A$. The fact that LE_B is empty implies $(1-x_B)/x_B \leq \eta^{1/\alpha}$ so we have $n_A/n_B < \eta^{1/\alpha}$ for all such n. This gives $u_A(n) < \eta u_B(n)$ in (7.11) for all such n. From D7.2(c) we obtain the final allocation n = (0, N).

In case (ii), the utility functions for peace in (7.10) give $u_A(n) = s_A n_A^{\alpha-1}$ and $u_B(n) = s_B n_B^{\alpha-1}$ for all n with $n_A \geq m_A$ and n_A/n_B below LE_P. From the construction of LE_P, this implies $\eta u_A(n) > u_B(n)$ for all such n. The allocation n with the smallest $n_A \geq m_A$ such that $\eta u_A(n) = u_B(n)$ is the one where n_A/n_B equals the lower bound of LE_P.

(c) Suppose LE_B is non-empty and consider two possibilities:

(i) m_A/m_B is below the lower bound of LE_B, which yields war;

(ii) m_A/m_B is between the upper bound of LE_B and the lower bound of LE_P, which may yield either war or peace.

In case (i), the argument is the same is in (b)(i) above, except that now $n_A/n_B < \eta^{1/\alpha}$ follows from the fact that all allocations under consideration have n_A/n_B below the lower bound of LE_B.

In case (ii), suppose $(1/\eta)^{1/\alpha} < m_A/m_B < (1-x_B)/x_B$, which yields war. The utility functions for war in (7.10) give $u_A(n) = [\phi(s_A, s_B)/N]n_A^\alpha$ and $u_B(n) = [\phi(s_A, s_B)/N]n_B^\alpha$ for all n with $n_A \geq m_A$ and $n_A/n_B < (1-x_B)/x_B$. From $(1/\eta)^{1/\alpha} < m_A/m_B \leq n_A/n_B$, at any such n we have $u_B(n) < \eta u_A(n)$. This includes the allocation m. From D7.2(b), no such n can be a final allocation. Now consider n with $n_A/n_B \geq (1-x_B)/x_B$ where n_A/n_B is below the lower bound of LE_P. Any such n yields peace. At $n_A/n_B = (1-x_B)/x_B$ the function $u_B(n)$ is continuous while $u_A(n)$ has an upward jump. This maintains $u_B(n) < \eta u_A(n)$. From the construction of LE_P, the allocation n with the smallest $n_A \geq m_A$ such that $u_B(n) = \eta u_A(n)$ is the one where n_A/n_B equals the lower bound of LE_P. If instead the initial allocation has $m_A/m_B \geq (1-x_B)/x_B$ and m_A/m_B is below the lower bound of LE_P, we repeat the last part of the argument above.

(d) The argument is symmetric to case (b) above.

(e) The argument is symmetric to case (c) above.

Proof of Lemma 7.2.

Fix $\sigma^t \in (0, \infty)$ and $m_A^t/m_B^t \in (0, \infty)$. We proceed in the following steps.

(a) Recall the definitions of $x_A(\sigma)$ and $x_B(\sigma)$ in (7.8). From Proposition 7.2, the pair $(\sigma^t, m_A^t/m_B^t)$ determines whether m_A^t/m_B^t is in one of the sets LE_B, LE_P, or LE_A. If it is, then from D7.2(a) we have $n_A^t/n_B^t = m_A^t/m_B^t$.

(b) If m_A^t/m_B^t is not in one of the sets LE_B, LE_P, or LE_A, this and the fact that m^t is interior implies that one of the four cases (b)-(e) in Proposition 7.3 applies. From Proposition 7.2, σ^t determines whether LE_B is empty or non-empty and likewise for LE_A. The ratios $(\sigma^t, m_A^t/m_B^t)$ together determine which of (b)-(e) in Proposition 7.3 applies, as well as the final allocation n^t, where n^t must be at the boundary $(N^t, 0)$, at the boundary $(0, N^t)$, the allocation associated with the lower bound of LE_P, or the allocation associated with the upper bound of LE_P.

(c) If step (a) applies with $m_A^t/m_B^t \in LE_P$, or step (b) applies and n^t is at the lower or upper bound of LE_P, there is peace in period t. This follows because all ratios in the LE_P interval obey the conditions for peace in Proposition 7.1 by construction. We then obtain m_A^{t+1}/m_B^{t+1} from (7.12), where the new allocation m^{t+1} is interior.

(d) If step (a) applies with $m_A^t/m_B^t \in LE_B$ or $m_A^t/m_B^t \in LE_A$, there is a non-trivial war in period t and we obtain m_A^{t+1}/m_B^{t+1} from (7.13), where m^{t+1} is interior. If step (b) applies and n^t is $(N^t, 0)$ or $(0, N^t)$, there is a trivial war in period t, and again we obtain m_A^{t+1}/m_B^{t+1} from (7.13), where m^{t+1} is interior.

Proof of Proposition 7.4 (war and peace with Malthusian dynamics).

From Proposition 7.3, a non-trivial war occurs in period t+1 iff $m_A^t/m_B^t \in LE_B$ or $m_A^t/m_B^t \in LE_A$. In all other cases, either $n^{t+1} = (N^{t+1}, 0)$ or $n^{t+1} = (0, N^{t+1})$ so there is a trivial war; or $n_A^{t+1}/n_B^{t+1} \in LE_P$ so there is peace.

(a) Proposition 7.2(a) shows that a necessary condition for $m_A^{t+1}/m_B^{t+1} \in LE_B$ is $\eta^{1/\alpha} \leq m_A^{t+1}/m_B^{t+1} \leq (1/\eta)^{1/\alpha}$. Proposition 7.2(c) shows that the same condition is necessary for $m_A^{t+1}/m_B^{t+1} \in LE_A$.

(b) Proposition 7.2(a) shows that a necessary condition for $m_A^{t+1}/m_B^{t+1} \in LE_B$ is $m_A^{t+1}/m_B^{t+1} < [1-x_B(\sigma^{t+1})]/x_B(\sigma^{t+1})$. Proposition 7.2(c) shows that a necessary condition for $m_A^{t+1}/m_B^{t+1} \in LE_A$ is $m_A^{t+1}/m_B^{t+1} > x_A(\sigma^{t+1})/[1-x_A(\sigma^{t+1})]$.

When the necessary condition in (a) is combined with the necessary condition for LE_B in (b), by Proposition 7.2 this suffices for $m_A^{t+1}/m_B^{t+1} \in LE_B$. The result for LE_A is the same.

We want to show that one of the inequalities in (b) holds iff $\sigma^{t+1} \notin [\sigma_A^{t+1}, \sigma_B^{t+1}]$ as in Proposition 7.4(b). The solutions for σ_A^{t+1} and σ_B^{t+1} exist and are unique due to the continuity of x_A and x_B; the monotonicity of these functions from (7.9b); and the limit properties of these functions from (7.9c) and (7.9d). We next show $\sigma_A^{t+1} < \sigma_B^{t+1}$. Suppose $\sigma_A^{t+1} = \sigma_B^{t+1}$. From (7.9a) we have $x_A(\sigma_A^{t+1}) + x_B(\sigma_B^{t+1}) > 1$, which contradicts the definition of σ_A^{t+1} and σ_B^{t+1}. Suppose $\sigma_A^{t+1} > \sigma_B^{t+1}$. From (7.9b), x_A is an increasing function, so this gives $x_A(\sigma_A^{t+1}) + x_B(\sigma_B^{t+1}) > x_A(\sigma_B^{t+1}) + x_B(\sigma_B^{t+1}) > 1$, which again contradicts the definition of σ_A^{t+1} and σ_B^{t+1}. Thus, $\sigma_A^{t+1} < \sigma_B^{t+1}$.

We now establish $\sigma_A^{t+1} < \sigma^t < \sigma_B^{t+1}$. Using the monotonicity of x_A and x_B in (7.9b), this holds iff $x_A(\sigma_A^{t+1}) < x_A(\sigma^t)$ and $1 - x_B(\sigma^t) < 1 - x_B(\sigma_B^{t+1})$. By the definitions of σ_A^{t+1} and σ_B^{t+1}, these inequalities hold iff $1 - x_B(\sigma^t) < m_A^{t+1}/N^{t+1} < x_A(\sigma^t)$. We will show that the latter pair of inequalities is always satisfied.

(i) Suppose there is a (trivial or non-trivial) war in period t. This implies $L_A^t/L_B^t = (\sigma^t)^{1/(1-\alpha)}$ where (L_A^t, L_B^t) is obtained from (7.5d). From (7.13), $m_A^{t+1}/m_B^{t+1} = (\sigma^t)^{1/(1-\alpha)}$. We know L_A^t/L_B^t equalizes average products at the productivity ratio σ^t so the same is true for m_A^{t+1}/m_B^{t+1}. Proposition 7.1 gives $1\text{-}x_B(\sigma^t) < m_A^{t+1}/N^{t+1} < x_A(\sigma^t)$.

(ii) Suppose there is peace in period t. From Proposition 7.3, this implies $n_A^t/n_B^t \in LE_P$. First consider the case in which $n_A^t/n_B^t > (\sigma^t)^{1/(1-\alpha)}$ so n_A^t/n_B^t exceeds the group size ratio that equalizes average products in period t. By (7.12), $m_A^{t+1}/m_B^{t+1} = \sigma^t(n_A^t/n_B^t)^\alpha$. This gives $n_A^t/n_B^t > m_A^{t+1}/m_B^{t+1} > (\sigma^t)^{1/(1-\alpha)}$. Due to $n_A^t/n_B^t \in LE_P$ and the fact that $(\sigma^t)^{1/(1-\alpha)}$ is in the interior of LE_P in period t, m_A^{t+1}/m_B^{t+1} is in the interior of the set LE_P defined by σ^t in period t. Proposition 7.2(b) then gives $1\text{-}x_B(\sigma^t) < m_A^{t+1}/N^{t+1} < x_A(\sigma^t)$. A parallel argument yields the same result for the case in which $n_A^t/n_B^t < (\sigma^t)^{1/(1-\alpha)}$. The only other case is $n_A^t/n_B^t = (\sigma^t)^{1/(1-\alpha)}$, which gives $m_A^{t+1}/m_B^{t+1} = (\sigma^t)^{1/(1-\alpha)}$. Again, m_A^{t+1}/m_B^{t+1} is in the interior of the set LE_P defined by σ^t and Proposition 7.2(b) gives $1\text{-}x_B(\sigma^t) < m_A^{t+1}/N^{t+1} < x_A(\sigma^t)$.

This concludes the proof that $\sigma_A^{t+1} < \sigma^t < \sigma_B^{t+1}$.

When $\sigma^{t+1} < \sigma_A^{t+1}$, the monotonicity of x_A gives $x_A(\sigma^{t+1}) < x_A(\sigma_A^{t+1}) \equiv m_A^{t+1}/N^{t+1}$ or $x_A(\sigma^{t+1})/[1\text{-}x_A(\sigma^{t+1})] < m_A^{t+1}/m_B^{t+1}$. When $\sigma^{t+1} > \sigma_B^{t+1}$, the monotonicity of x_B gives 1-

$x_B(\sigma^{t+1}) > 1- x_B(\sigma_B^{t+1}) \equiv m_A^{t+1}/N^{t+1}$ or $[1-x_B(\sigma^{t+1})]/x_B(\sigma^{t+1}) > m_A^{t+1}/m_B^{t+1}$. When $\sigma_A^{t+1} \leq \sigma^{t+1} \leq \sigma_B^{t+1}$, we have $[1-x_B(\sigma^{t+1})]/x_B(\sigma^{t+1}) \leq m_A^{t+1}/m_B^{t+1} \leq x_A(\sigma^{t+1})/[1-x_A(\sigma^{t+1})]$. Thus, one of the inequalities in (b) of the proof holds iff $\sigma^{t+1} \notin [\sigma_A^{t+1}, \sigma_B^{t+1}]$ as in Proposition 7.4(b).

Proof of Corollary to Proposition 7.4.

From Proposition 7.4, $\sigma_A^{t+1} < \sigma^t = \sigma^{t+1} < \sigma_B^{t+1}$ implies that there cannot be a non-trivial war in period t+1 regardless of whether there is war or peace in period t. A trivial war can be ruled out using (i) and (ii) in the proof of Proposition 7.4 and substituting $\sigma^t = \sigma^{t+1}$ to show that $m_A^{t+1}/m_B^{t+1} \in LE_P$ for period t+1. Proposition 7.3(a) then yields peace in period t+1.

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Chapter 8: Warfare Between Elite Groups

Proofs of Formal Propositions

Proof of Lemma 8.1.

(a) From Table 8.1, F is a dominant strategy for elite i iff $we_i \geq R - wm_i$ or $m_i \geq R/w - e_i$. When this weak inequality holds, F is a best reply to F, a unique best reply to A (using $p_i < 1$, which is true for all $m_i \geq 0$ given $m_j > 0$), and a unique best reply to D (again using $p_i < 1$). When $we_i < R - wm_i$ and $j \neq i$ uses F, elite i is strictly better off using A rather than F, so F is not a dominant strategy for elite i.

(b) Suppose F is used by elite $j \neq i$. F is a best reply for elite i iff $we_i \geq R - wm_i$ as in part (a), which implies that F is dominant for i. A is a best reply for i iff $we_i \leq R - wm_i$. D is never a best reply for i because $r_i - wm_i < R - wm_i$ always holds.

Proof of Lemma 8.2.

(a) From Table 8.1, a necessary condition for A to be a dominant strategy for elite i is $p_iR - wm_i \geq \max \{r_i - wm_i, we_i\}$ so A is a best reply to D. We will show that the same condition is likewise sufficient for A to be dominant. By construction this condition implies that A is a best reply to D. Because $p_iR - wm_i \geq we_i$ it implies that A is a best reply to A. Furthermore, A is a unique best reply to F because $r_i - wm_i < R - wm_i$ (so A is strictly preferred to D) and by assumption $we_i < R - wm_i$ (so A is strictly preferred to F). Hence, $p_iR - wm_i \geq \max \{r_i - wm_i, we_i\}$ is both necessary and sufficient for A to be a dominant strategy for elite i.

(b) Immediate from Table 8.1.

Proof of Proposition 8.1 (the combat stage).

(a) Immediate from Lemma 8.1.

(b) From Lemma 8.1, F is a dominant strategy for i but not for $j \neq i$. By A8.1 elite i uses F. From Table 8.1, the unique best reply for $j \neq i$ is A because $R - wm_j > r_j - wm_j$ always holds and $R - wm_j > we_j$ holds by assumption. Hence, FA is the only Nash equilibrium.

(c) Immediate from Lemma 8.2.

(d) From Lemma 8.2, A is a dominant strategy for elite i but not for elite $j \neq i$. By A8.1 elite i uses A.

(i) If $we_j < p_jR - wm_j$ then from Table 8.1, A and D are both best replies for j while F is not. Hence, the only Nash equilibria are {AA, AD}.

(ii) If $we_j = p_jR - wm_j$ then from Table 8.1, A, D, and F are all best replies for j. Hence, the only Nash equilibria are {AA, AD, AF}.

(iii) If $we_j > p_jR - wm_j$ then from Table 8.1, the unique best reply for j is F. Hence, the only Nash equilibrium is AF.

(e) First, we show that $we_i < p_iR - wm_i$ cannot hold for both i = A, B. Suppose it did. By assumption, $\max \{we_i, r_i - wm_i\} > p_iR - wm_i$ for both i = A, B. Together these imply $r_A > p_AR$ and $r_B > p_BR$. Summation gives $R > R$, which is a contradiction. Suppose instead $we_i < p_iR - wm_i$ for elite i and $we_j \geq p_jR - wm_j$ for $j \neq i$. The first of these inequalities gives $we_i < p_iR - wm_i < r_i - wm_i$. In order to avoid the same contradiction as before we must then have $r_j - wm_j < p_jR - wm_j < we_j$. These two series of inequalities, along with $r_i - wm_i < R - wm_i$, imply from Table 8.1 that AF is a Nash equilibrium. Elite i never uses F, so in any other Nash equilibrium elite i must either use A or D. If i uses D then j's unique best reply is F. However, i's unique best reply to F is A, so there cannot be a Nash equilibrium where i uses D. The only remaining possibility is that i uses A, but then j's unique best reply is F. This shows that AF is the only Nash equilibrium.

(f) We are assuming max $\{we_i, r_i - wm_i\} > p_iR - wm_i$ for both i = A, B and $we_i \geq p_iR - wm_i$ for both i = A, B. The fact that $we_i \geq p_iR - wm_i$ implies from Table 8.1 that F is a best reply to A for elite i. It is always true that A is a unique best reply to F for $j \neq i$ because $R - wm_j > r_j - wm_j$ and $R - wm_j > we_j$ (where the latter follows from the assumption that F is not a dominant strategy for either elite). Therefore, FA is a Nash equilibrium, and by a symmetric argument so is AF. There is no other Nash equilibrium where i uses F, so in any other Nash equilibrium i must use A or D. By symmetry, in any other Nash equilibrium $j \neq i$ must also use A or D. Suppose there is some other Nash equilibrium of this kind. Next observe that we cannot have $we_i = p_iR - wm_i$ for both i = A, B. This would imply $r_i - wm_i > p_iR - wm_i$ for both i = A, B and therefore $r_i > p_iR$ for both i = A, B. Summation then gives R > R, which is a contradiction. Thus, we must have $we_i > p_iR - wm_i$ for at least one of i = A, B. The elite i has F as a unique best reply to A. But we already know that in any additional Nash equilibrium, each elite must use either A or D. This eliminates the possibility that j uses A. The only remaining possibility is that j uses D. But A is not a best reply to D for i due to $we_i > p_iR - wm_i$, and because i uses either A or D, i must use D. Hence, we need to know whether DD is a Nash equilibrium. If it is, then from Table 8.1 we have $r_i - wm_i \geq p_iR - wm_i$ for both i = A, B. To avoid the same contradiction as before, we need $r_i - wm_i =$

$p_iR - wm_i$ for both i = A, B. This implies $we_i > p_iR - wm_i = r_i - wm_i$ for both i = A, B. But then the unique best reply to D is F for both i = A, B, so DD is not a Nash equilibrium. This establishes that there are no Nash equilibria other than {AF, FA}.

With regard to DF and FD, use Table 8.1 and suppose elite j chooses F. Because $r_i < R$, elite i always has a strictly higher payoff by responding with A rather than D. The roles of i and $j \neq i$ can be interchanged in this argument. Hence, neither DF nor FD can be a Nash equilibrium. The reasons for the exclusion of DD are discussed in the text.

Proof of Lemma 8.3.

Conditions (a)-(d) for AF and FA to both be Nash equilibria come from Table 8.1. The remarks about the relationship of these conditions to Figure 8.2 are straightforward. In a mixed strategy Nash equilibrium, λ_j is chosen to make elite i indifferent between A and F. Using Table 8.1, the payoff from A for elite i is $\lambda_j(p_iR - wm_i) + (1-\lambda_j)(R - wm_i)$. The payoff from F for elite i is we_i for all values of $\lambda_j \in [0, 1]$. Equating these payoffs gives $\lambda_j = (R/w - m_i - e_i)/(R/w)p_j$ and similarly $\lambda_i = (R/w - m_j - e_j)/(R/w)p_i$ where $p_i > 0$ and $p_j > 0$ because we are assuming $(m_i, m_j) > 0$. Elite i attacks with positive probability when $\lambda_i > 0$, which is true iff condition (a) in Lemma 8.3 holds strictly. Elite i flees with positive probability when $1-\lambda_i > 0$, which is true iff condition (c) holds strictly. In this case elite i strictly prefers flight to warfare, and graphically (m_i, m_j) is above the IC_i^{WF} curve in Figure 8.2. Parallel results are obtained for elite j by interchanging subscripts. The expected payoff for elite i is $\lambda_i[\lambda_j(p_iR - wm_i) + (1-\lambda_j)(R - wm_i)] + (1-\lambda_i)we_i$. But λ_j was constructed to equate the expression in brackets to we_i and therefore elite i's payoff in equilibrium is also we_i. Similarly, elite j's equilibrium payoff is we_j.

Proof of Lemma 8.4.

Consider a recruitment equilibrium $(m_i, m_j) > 0$ in the interior of the AA region. This must have the property that m_i is a local maximizer of $p_iR - wm_i$ for the given m_j and m_j is a local maximizer of $p_jR - wm_j$ for the given m_i. Using $p_i = m_i/(m_i + m_j)$, the first order condition to maximize i's payoff is $R/(m_i + m_j) - m_iR/(m_i + m_j)^2 - w = 0$. The second derivative is negative so elite i's payoff is strictly concave in m_i and the first order condition is sufficient for a maximum. For elite j we have the first order condition $R/(m_i + m_j) - m_iR/(m_i + m_j)^2 - w = 0$, and again the payoff function is strictly concave. Writing $m_i + m_j = M$, the first order conditions reduce to $m_iR = wM^2$ and $m_jR = wM^2$. Summing these gives $MR = 2wM^2$ or $M = R/2w$. This yields $m_i = m_j = R/4w$.

The point $(m_i, m_j) = (R/4w, R/4w)$ is in the interior of the AA region iff $p_iR - wm_i > we_i$ and $p_jR - wm_j > we_j$ so that each elite strictly prefers warfare to flight. Substituting $(R/4w, R/4w)$ into these inequalities gives $R/4w > e_A$ and $R/4w > e_B$. In the rest of the proof, we assume these inequalities hold.

To see that $(R/4w, R/4w)$ is not a recruitment equilibrium, suppose elite i uses $m_i = R/4w$. If elite j also uses $m_j = R/4w$ then elite j's payoff is $R/4$. Suppose instead that j deviates to the army size m_j' that makes elite i indifferent between warfare and flight. This point is on IC_i^{WF} along the upper boundary of the AA region adjacent to the FA region. The level of m_j' in this deviation is obtained from $m_iR/(m_i + m_j') - wm_i = we_i$ where $m_i = R/4w$. This yields $m_j' = (R/4w)(3R/4 - we_i)/(R/4 + we_i)$. Substituting m_j' into elite j's grabbing payoff $R - wm_j$ shows that $R - wm_j'$ exceeds $R/4$. However, along the boundary between AA and FA, the outcome is AA due to A8.1. But increasing the size of elite j's army by a small additional amount $\varepsilon > 0$ moves the outcome from AA to FA and achieves the grabbing payoff R - $w(m_j' + \varepsilon)$. This also exceeds R/4 when $\varepsilon > 0$ is small enough, and therefore (R/4w, R/4w) is not a recruitment equilibrium. A parallel argument shows that elite i has profitable deviations to the AF region, slightly to the right of the lower boundary for the AA region.

Proof of Lemma 8.5.

The equation describing IC_i^{WF} is $p_iR - wm_i = we_i$ and the equation describing IC_j^{WF} is $p_jR - wm_j = we_j$. Rearranging these gives $m_j = \phi_i(m_i) = [m_i/(m_i + e_i)](R/w - e_i - m_i)$ and $m_i = \phi_j(m_j) = [m_j/(m_j + e_j)](R/w - e_j - m_j)$ respectively. Suppose (m_i, m_j) is on both curves. Writing $m_i + m_j = M$ gives $m_iR = Mw(m_i + e_i)$ and $m_jR = Mw(m_J + e_j)$. Summation yields $MR = Mw(M + e_i + e_j)$ or $M = R/w - e_i - e_j$. More algebra gives $m_i = [e_i/(e_i + e_j)](R/w - e_i - e_j)$ and $m_j = [e_j/(e_j + e_i)](R/w - e_i - e_j)$ where both are positive because $R/w > e_i + e_j$ from (8.3). This is the unique positive intersection point in (8.11).

(a) Next differentiate $m_j = \phi_i(m_i)$ to get $\phi_i'(m_i) = (R/w - e_i - m_i)[1/(m_i + m_j) - m_i/(m_i + e_i)^2] - m_i/(m_i + e_i)$. We are interested in the sign of this derivative at the intersection point from the preceding paragraph. Substituting for m_i shows that $\phi_i'(m_i)$ is strictly positive at the intersection point iff $R/w < (e_i + e_j)^2/e_i$. Similarly, $\phi_j'(m_j)$ is strictly positive at the intersection point iff $R/w < (e_i + e_j)^2/e_j$. Therefore, both derivatives are strictly positive at the intersection, and both IC_i^{WF} and IC_j^{WF} are rising, iff $R/w < (e_i + e_j)^2 / \max(e_i, e_j)$. The result in Lemma 8.5(a) about the lower bound for R/w from (8.3) involves easy algebra.

(b) Let $e_j < e_i$. From previous results, $\phi_j'(m_j)$ is strictly positive at the intersection point, and IC_j^{WF} is rising, iff $R/w < (e_i + e_j)^2/e_j$. On the other hand, $\phi_i'(m_i)$ is strictly negative at the intersection point, and IC_i^{WF} is falling, iff $R/w > (e_i + e_j)^2/e_i$. These two inequalities give the result in Lemma 8.5(b). The roles of e_i and e_j can be interchanged in the preceding arguments.

(c) Both $\phi_i'(m_i)$ and $\phi_j'(m_j)$ are strictly negative at the intersection point, and IC_i^{WF} and IC_j^{WF} are both falling, iff $R/w > (e_i + e_j)^2/e_i$ and $R/w > (e_i + e_j)^2/e_j$ both hold. This gives the result in Lemma 8.5(c).

Proof of Proposition 8.2 (the recruitment stage).

The proof is organized into three parts corresponding to parts (a), (b), and (c) of Lemma 8.5. We rely heavily on the graphs in Figures 8.3, 8.4, and 8.5. We are limiting attention to points $(m_i, m_j) > 0$ such that $m_i < R/w - e_i$ and $m_j < R/w - e_j$. The latter two inequalities imply that grabbing is always strictly preferred to flight, where flight yields we_i for elite i or we_j for elite j. We will use A8.1, A8.2, and A8.3, plus the conditions in Remark 8.1 ensuring that the point P is below $R/w - e_j$ and the point Q is to the left of $R/w - e_i$.

(a) Low stratification (Lemma 8.5(a), Figure 8.3).

First, we consider the case in Figure 8.3 where Q is above the dashed horizontal line through P, and P is to the right of the dashed vertical line through Q. Other cases are addressed at the end of part (a) of the proof. We examine points in the four regions of the graph: mixing, FA, AF, and AA.

Mixing. Points in the interior of the shaded rectangle are both above the dashed horizontal line through P and to the right of the dashed vertical line through Q. At such a point, elite i has the payoff we_i. At any other point on the same horizontal line, elite i has the same payoff through mixing or FA. Elite j has the payoff we_j. At any other point on the same vertical line, elite j has the same payoff through mixing or AF. Hence, neither elite has a profitable deviation, and the point is a Nash equilibrium.

At any point in this set, each elite attacks with positive probability because strict inequalities hold in (a) and (b) of Lemma 8.3. Each elite flees with positive probability because strict inequalities hold in (c) and (d) of Lemma 8.3.

All other points in the mixing region are (i) on or to the left of the dashed vertical line through Q, or (ii) on or below the dashed horizontal line through P, or both. For any point of type (i), elite j can deviate up or down to reach a point in the FA region, resulting in the payoff R - wm_j, which is strictly larger than the mixing payoff we_j. For any point of type (ii), elite i can deviate left or right to reach a point in the AF region, resulting in the payoff R - wm_i, which is strictly larger than the mixing payoff we_i. This shows that no other point in the mixing region is a Nash equilibrium.

FA. Recall that the FA region does not include the lower boundary with the AA region; such boundary points are part of AA. At any point in FA where elite j can deviate down and still remain in the FA region, we do not have a Nash equilibrium, because such a deviation enables elite j to grab at lower cost. This rules out all points in FA except those on the boundary segment NQ (excluding N, which is in AA). However, for the points along NT that are on or below the dashed horizontal line through P, elite i can go from a payoff of we_i to the higher grabbing payoff R - wm_i by deviating to the right to a point in AF. This leaves points along the segment TQ (excluding T). At any such point, elite j achieves the grabbing payoff R - wm_i. Elite j cannot gain by deviating up or down because this either leads to grabbing at a higher cost within FA, or the lower payoff we_j through mixing or AF. Also at any such point, elite i has the payoff we_i. Elite i cannot gain by deviating to the left or right because this gives the same payoff we_i via mixing or FA. Thus, points on the segment TQ (excluding T) are Nash equilibria.

AF. Recall that the AF region does not include the left boundary with the AA region; such boundary points are part of AA. A parallel argument to the one used in the preceding paragraph shows that no point in AF where elite i can deviate left and still be in the AF region can be a Nash equilibrium. This rules out all points except those on the boundary segment NP (excluding N, which is in AA). However, for points along NS elite j can go from a payoff of we_j to the higher grabbing payoff R - wm_j by deviating up to a point in FA. This leaves points on the segment SP (excluding S). It can be shown by similar arguments to those in the preceding paragraph that these are Nash equilibria.

AA. Recall that the AA region includes its boundary. Along the upper boundary, elite j can exploit a discontinuity in the payoff function by raising m_j slightly and jumping from the AA payoff p_jR - wm_j to the FA payoff R - wm_j. Such a deviation is profitable. Along the lower boundary, elite i can exploit a similar discontinuity by raising m_i slightly and jumping from p_iR - wm_i to the AF payoff R - wm_i. Such a deviation is profitable. This rules out all points on the boundary of AA. Lemma 8.4 shows that there cannot be any Nash equilibria in the interior of the AA region. Thus, the only Nash equilibria in Figure 8.3 are those already described.

As explained in the text, it is mathematically possible (contrary to Figure 8.3) to have Q on or below the dashed horizontal line through P, or to have P on or to the left of the dashed vertical line through Q. It is not possible to have both cases simultaneously. In the former case, points on the segment TQ are excluded because elite i can deviate to a point in AF and get the grabbing payoff R - wm_i rather than we_i. In the latter case, points on the segment SP are excluded because elite j can deviate to a point in FA and get the grabbing payoff R - wm_j rather than we_j. The other arguments go through unchanged.

(b) Intermediate stratification (Lemma 8.5(b), Figure 8.4).

First, we consider the case in Figure 8.4 where Q is above the dashed horizontal line through P. Other cases will be addressed at the end of part (b) of the proof. We examine points in the four regions of the graph: mixing, FA, AF, and AA.

Mixing. Points in the interior of the shaded rectangle or on the heavy segment of its lower boundary are both (i) to the right of the dashed vertical line through Q and (ii) on or above the dashed horizontal line through P. At such a point, elite j has the payoff we_j. At any other point on the same vertical line, elite j achieves the same payoff through mixing or AF. Elite i has the payoff we_i. At any other point on the same horizontal line, elite i has the same payoff through mixing or FA (or in the case of point P, through AA). Hence, neither elite has a profitable deviation, and the point is a Nash equilibrium.

At any point in this set, each elite attacks with positive probability because strict inequalities hold in conditions (a) and (b) of Lemma 8.3. Each elite flees with positive probability because strict inequalities hold in conditions (c) and (d) of Lemma 8.3.

All other points in the mixing region are either (i) on or to the left of the dashed vertical line through Q or (ii) below the dashed horizontal line through P (or both). For any point of type (i), elite j can deviate up or down to reach a point in the FA region, resulting in the payoff $R - wm_j$, which is strictly larger than the mixing payoff we_j. For any point of type (ii), elite i can deviate leftward to reach a point in the AF region or the interior of the AA region, either of which yields a higher payoff than the mixing payoff we_i. This shows that no other point in the mixing region is a Nash equilibrium.

FA. Recall that the FA region does not include the lower boundary with the AA region; such boundary points are part of AA. At any point in FA where elite j can deviate down and still remain in the FA region, we do not have a Nash equilibrium, because such a deviation enables elite j to grab at lower cost. This rules out all points in FA except those on the boundary segment NQ (excluding N, which is in AA). For the points along NT that are below the dashed horizontal line through P, elite i can switch from a payoff of we_i to the higher warfare payoff p_iR - wm_i by deviating leftward to some point in the interior of AA. This leaves the points along the segment TQ. At any such point, elite j obtains a grabbing payoff R - wm_i. Elite j cannot gain by deviating up or down because this either results in grabbing at a higher cost within FA, or the lower payoff we_j through mixing or AF. Also at any such point, elite i has the payoff we_i. Elite i cannot gain by deviating left or right because this gives the same payoff we_i through mixing or FA (or in the case of point P, through AA). Thus, points on the segment TQ are Nash equilibria.

AF. Recall that the AF region does not include the left boundary with the AA region; such boundary points are part of AA. No point in AF can be a Nash equilibrium if elite i can deviate left and still be in the AF region, because this enables elite i to grab at a lower cost. This rules out all points in the AF region.

AA. Recall that the AA region includes its boundary. Along the upper boundary (including point N), elite j can exploit a discontinuity in its payoff function by raising m_j slightly and jumping from the AA payoff p_jR - wm_j to the FA payoff R - wm_j. Such a deviation is profitable. Along the lower boundary, elite i can raise m_i slightly and jump from the AA payoff p_iR - wm_i to the AF payoff R - wm_i. Such a deviation is profitable. This rules out all the points on the boundary of AA. Lemma 8.4 shows that there cannot be any Nash equilibria in the interior of the AA region. Thus, the only Nash equilibria in Figure 8.4 are those already described.

As explained in the text, it is mathematically possible (contrary to Figure 8.4) to have Q on or below the dashed horizontal line through P. When Q is below this line, all points on the segment TQ are excluded because elite i can deviate left to a point in the interior of the AA region and get a higher payoff than we_i. When Q is on this line, Q is still an equilibrium because elite i obtains the payoff we_i from all left or right deviations, including P where elite i gets this payoff from AA. All other points on TQ are excluded for the same reason as before. The other arguments in the proof go through unchanged.

(c) High stratification (Lemma 8.5(c), Figure 8.5).

We examine points in the four regions of the graph: mixing, FA, AF, and AA.

Mixing. Points in the shaded rectangle (including the heavy lower and leftward boundaries) are both (i) on or to the right of the dashed vertical line through Q and (ii) on or above the dashed horizontal line through P. At such a point, elite j has the payoff we_j. At any other point on the same vertical line, elite j gets the same payoff through mixing or AF (or in the case of point Q, through AA). Elite i has the payoff we_i. At any other point on the same horizontal line, elite i has the same payoff through mixing or FA (or in the case of point P, through AA). Hence, neither elite has a profitable deviation, and the point is a Nash equilibrium.

At any point in this set, each elite attacks with positive probability because strict inequalities hold in conditions (a) and (b) of Lemma 8.3. Each elite flees with positive probability because strict inequalities hold in conditions (c) and (d) of Lemma 8.3.

All other points in the mixing region are either (i) to the left of the dashed vertical line through Q or (ii) below the dashed horizontal line through P (or both). For any point of type (i), elite j can deviate down to a point in the interior of the AA region and get the warfare payoff p_jR - wm_j, which is strictly larger than the mixing payoff we_j. For any point of type (ii), elite i can deviate leftward to a point in the interior of the AA region and get the warfare payoff p_iR - wm_i, which is strictly larger than the mixing payoff we_i. This shows that no other point in the mixing region is a Nash equilibrium.

FA. Recall that the FA region does not include the lower boundary with the AA region; such boundary points are part of AA. At any point in FA where elite j can deviate down and still remain in the FA region, we do not have a Nash equilibrium, because such a deviation enables elite j to grab at a lower cost. This rules out all points in FA.

AF. Recall that the AF region does not include the left boundary with the AA region; such boundary points are part of AA. At any point in AF where elite i can deviate left and still remain in the AF region, we do not have a Nash equilibrium, because such a deviation enables elite i to grab at a lower cost. This rules out all points in AF.

AA. Recall that the AA region includes its boundary. Along the upper boundary (except at point N), elite j can exploit a discontinuity in its payoff function by raising m_j slightly and jumping from the AA payoff p_jR - wm_j to the FA payoff R - wm_j. Such a deviation is profitable. Along the lower boundary (except at point N), elite i can likewise raise m_i slightly and jump from the AA payoff p_iR - wm_i to the AF payoff R - wm_i. Such a deviation is also profitable. This rules out all points on the boundary of AA except for N, where the payoffs are (we_i, we_j) because each elite is indifferent between warfare and flight. At this point, elite i can move left to the interior of AA and elite j can move down to the interior of AA. Each deviation is profitable. Lemma 8.4 shows that there cannot be any Nash equilibria in the interior of the AA region. Thus, the only Nash equilibria in Figure 8.5 are those already described.

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Chapter 10: Mesopotamian City-States: A Formal Model

Proofs of Formal Propositions

Proof of Proposition 10.1 (stratification).

From the solution for C following D10.1, $C/Z < e$ holds in equilibrium iff $N - e < e\beta(\theta)/\theta^{1/(1-\alpha)}$. From (10.6), $R(w)/e \geq w$ holds in equilibrium iff $e\beta(\theta)/(1-\alpha)\alpha^{\alpha/(1-\alpha)} \leq N - e$. There is a value of $N - e$ satisfying both conditions simultaneously iff $\theta < \alpha^{\alpha}(1-\alpha)^{1-\alpha} \equiv \theta_{max}$ as in (10.10). When this is true, the earlier inequalities ensure that both stratification constraints hold iff (10.11) holds. The lower bound $e\beta(\theta)/(1-\alpha)\alpha^{\alpha/(1-\alpha)}$ is increasing in θ due to the fact that $\beta(\theta)$ is increasing in θ. The upper bound $e\beta(\theta)/\theta^{1/(1-\alpha)}$ is decreasing in θ due to the definition of $\beta(\theta)$.

Comment 10.1. In the paragraph after (10.12), we claim the constraint $y \geq 0$ is satisfied for all agents in equilibrium. Consider a typical commoner with food consumption $y^C = x^C - pm$ where x^C is income and pm is expenditure on manufactured goods. The income of a commoner in food units is the wage w. In equilibrium all of the agents (both the elite and commoners) have $m = M/N$ where M is total manufactured output. We want to be assured that $y^C = w - pM/N \geq 0$, or $wN \geq pM$. But in a zero-profit equilibrium we will have $wL = pM$ as in equation (10.15), where L is the aggregate input of manufacturing labor. Because $A + L + e = N$ from labor market clearing, we have $L < N$. Hence, $wL = pM$ implies $wN > pM$, which implies $y^C > 0$. Individual elite and commoner agents have an identical expenditure pm on manufactured goods, but the elite agents are at least as well off as the commoners in any equilibrium that satisfies the stratification constraints. Thus, $x^E \geq x^C$ and $y^E \geq y^C > 0$.

Proof of Lemma 10.1.

From (10.12) and (10.14), $\phi(0) = qr$. Also, $\phi(L) = (N/L)[qM(L)/N]e^{-qM(L)/N}$ and the fact that $M(L) \to \infty$ as $L \to \infty$ imply $\phi(L) \to 0$ as $L \to \infty$.

The function $\phi(L)$ has a negative derivative at $L > 0$ iff the function $\ln \phi(L)$ has a negative derivative at $L > 0$. The latter function is

$$\ln \phi(L) = \ln q - qM(L)/N + \ln (e^{rL} - 1) - \ln L$$

so $$d[\ln \phi(L)]/dL = -(qr/N)e^{rL} + re^{rL}/(e^{rL} - 1) - 1/L \leq 0 \qquad \text{for } L > 0$$

iff $$= -(qrL/N)e^{rL} + rLe^{rL}/(e^{rL} - 1) - 1 \leq 0 \qquad \text{for } L > 0.$$

For compactness of notation, define $K \equiv rL$ so the last inequality becomes

$$-(qK/N)e^{K} + Ke^{K}/(e^{K} - 1) - 1 \leq 0 \qquad \text{for } K > 0.$$

Some reorganization gives

$$1 \leq e^{K}\{(qK/N)e^{K} - [K(q/N + 1) - 1]\} \qquad \text{for } K > 0$$

where the equality holds at $K = 0$. The derivative of the right side above is $e^{K}\varphi(K)$ where $\varphi(K) \equiv (q/N)(2K + 1)e^{K} - K(q/N + 1) - q/N$. Note that $\varphi(0) = 0$. We have

$$\varphi'(K) = (q/N)(3 + 2K)e^{K} - (q/N + 1)$$

so $\varphi'(0) = 2q/N - 1$. When A10.1 holds so that $q/N \geq 1/2$, we have $\varphi'(0) \geq 0$ and $\varphi'(K) > 0$ for all $K > 0$. This implies $\varphi(K) > 0$ for all $K > 0$. It follows that $e^{K}\varphi(K) > 0$ for all $K > 0$ and therefore $1 < e^{K}\{(qK/N)e^{K} - [K(q/N + 1) - 1]\}$ for all $K > 0$. In turn, this implies

$$d[\ln \phi(L)]/dL = -(qr/N)e^{rL} + re^{rL}/(e^{rL} - 1) - 1/L < 0 \quad \text{for all } L > 0$$

so $\phi(L)$ has a negative derivative at all $L > 0$. This completes the proof of the assertion in Lemma 10.1 that if A10.1 holds, $\phi(L)$ is decreasing for all $L > 0$.

Now suppose that A10.1 does not hold. This implies $\varphi'(0) < 0$. However, $\varphi'(K)$ is continuous, increasing in K, and has $\varphi'(K) \to \infty$ as $K \to \infty$. Thus, there is a unique $K_a > 0$ such that $\varphi(K)$ is falling on $(0, K_a)$ and rising on (K_a, ∞). From $\varphi(0) = 0$ and $\varphi(K) \to \infty$ as $K \to \infty$, there is a unique $K_b > K_a$ such that $\varphi(K)$ is negative on $(0, K_b)$ and positive on (K_b, ∞). This implies that $e^K\{(qK/N)e^K - [K(q/N + 1) - 1]\}$ is decreasing on $(0, K_b)$ and increasing on (K_b, ∞). This expression is equal to unity at $K = 0$ and goes to infinity as $K \to \infty$. From continuity it follows that there is a unique $K_c > K_b$ such that

$$e^K\{(qK/N)e^K - [K(q/N + 1) - 1]\} < 1 \qquad \text{for } 0 < K < K_c \qquad \text{and}$$

$$e^K\{(qK/N)e^K - [K(q/N + 1) - 1]\} > 1 \qquad \text{for } K > K_c.$$

From the definition of K and earlier results, this implies that there is some $L_c = K_c/r > 0$ such that

$$d[\ln \phi(L)]/dL = -(qr/N)e^{rL} + re^{rL}/(e^{rL} - 1) - 1/L > 0 \quad \text{for } 0 < L < L_c \qquad \text{and}$$

$$d[\ln \phi(L)]/dL = -(qr/N)e^{rL} + re^{rL}/(e^{rL} - 1) - 1/L < 0 \quad \text{for } L > L_c.$$

This completes the proof of the assertion in Lemma 10.1 that if A10.1 does not hold, there is some $L_c > 0$ such that $\phi(L)$ is increasing for $L < L_c$ and decreasing for $L > L_c$.

Proof of Proposition 10.2 (zero-profit equilibrium).

(a) Consider any zero-profit equilibrium as in D10.3 with $L^0 = 0$. We have such an equilibrium iff $p^0 \geq b'(0) = q$, $w^0 \geq p^0 M'(0) = p^0 r$, and $A^0 = N - e = A(w^0, \theta) = \beta(\theta)/(w^0)^{1/(1-\alpha)}$. From (10.9), the latter equalities uniquely determine the wage, with $w^0 = [\beta(\theta)/(N-e)]^{1-\alpha}$. The wage w^0 is the same as defined in the agricultural equilibrium from D10.1. The conditions $p^0 \geq q$ and $w^0 \geq p^0 r$ are both satisfied iff $[\beta(\theta)/(N-e)]^{1-\alpha}/r \geq p^0 \geq q$. There is such a p^0 iff $[\beta(\theta)/(N-e)]^{1-\alpha} \geq qr$. By (10.18), the latter inequality holds iff $\theta_0 \leq \theta$.

(b) Consider any zero-profit equilibrium as in D10.3 with $L^0 > 0$. We have such an equilibrium iff

(b1) $p^0 = b'[M(L^0)/N]$

(b2) $w^0 = p^0 M(L^0)/L^0$

(b3) $A^0 = A(w^0, \theta) = \beta(\theta)/(w^0)^{1/(1-\alpha)}$

(b4) $A^0 + L^0 + e = N$

As in (10.16), (10.17), and Lemma 10.1, these imply $\beta(\theta)^{1-\alpha} = (N - e - L^0)^{1-\alpha}\phi(L^0)$, so this equality is necessary for an equilibrium with $L^0 > 0$. Conversely if $L^0 > 0$ satisfies this equality, there are (p^0, w^0, A^0) from (b1), (b2), (b3), and (b4) that constitute a zero-profit equilibrium as in D10.3. From Lemma 10.1, $\phi(0) = qr$, $\phi(\infty) = 0$, and $\phi'(L) < 0$ for all $L > 0$ due to A10.1. Also, $(N - e - L)^{1-\alpha} = (N - e)^{1-\alpha}$ at $L = 0$, $(N - e - L)^{1-\alpha} = 0$ at $L = N - e$, and $(N - e - L)^{1-\alpha}$ is decreasing for $L \in (0, N - e)$. By continuity and monotonicity there is a unique $L^0 > 0$ such that $\beta(\theta)^{1-\alpha} = (N - e - L^0)^{1-\alpha}\phi(L^0)$ iff $\beta(\theta)^{1-\alpha} < (N - e)^{1-\alpha}\phi(0)$ or equivalently $\beta(\theta)^{1-\alpha} < qr(N - e)^{1-\alpha}$.

From (10.18) the latter inequality holds iff $\theta < \theta_0$. The uniqueness of (p^0, w^0, A^0) follows from (b1), (b2), (b3), and (b4).

(i) Differentiability of L^0 as a function of θ follows from the differentiability of $\phi(L^0)$ and the implicit function theorem. Differentiability of (p^0, w^0, A^0) as functions of θ follows from the fact that these are differentiable functions of L^0.

(ii) A10.1, Lemma 10.1, and the resulting condition $\phi'(L^0) < 0$ for all $L^0 > 0$ ensure that L^0 is inversely related to β and therefore to θ. The equilibrium conditions (b1) and (b4) ensure that p^0 and A^0 are inversely related to L^0 and therefore directly related to θ. A10.1, Lemma 10.1, (b1), and (b2) ensure that w^0 is inversely related to L^0 and therefore directly related to θ.

(iii) The equation $\beta(\theta_0)^{1-\alpha} = qr(N - e)^{1-\alpha}$ from (10.18) implies $L^0 \to 0$ as $\theta \to \theta_0$. The other limits follow from the equilibrium conditions (b1), (b2), (b3), and (b4). These limits are identical to the solutions from part (a) at θ_0, so the equilibrium values are continuous at θ_0.

(iv) Let $\theta = 0$ so $\beta(0) \equiv \alpha^{1/(1-\alpha)}$ in (10.17). We assume $\theta_0 > 0$ in part (b) of Proposition 10.2 and equilibrium requires $\beta(0)^{1-\alpha} = \alpha = (N - e - L^0)^{1-\alpha}\phi(L^0)$. To show that $L^0 > 0$, suppose instead $L^0 = 0$. Using (10.18) this implies $\theta_0 = 0$, a contradiction. We have $p^0 > 0$ from (b1), $w^0 > 0$ from (b2), and $A^0 > 0$ from (b3).

Comment 10.2. In comment 10.1, we showed that commoners always have positive food consumption ($y^C > 0$) in a zero-profit equilibrium. This is harder to prove for an elite taxation equilibrium. We want $x^C - pm = y^C > 0$ from the commoner budget constraint, where commoner income is the wage: $w = x^C$. Combining this with the fact that in any equilibrium we have $m = M/N$, we want $w > pM/N$ so that the wage exceeds per capita expenditure on manufactured goods. It is difficult to establish such a result in general. However, it will be shown later that there is a threshold θ_e such that the elite allows $M > 0$ when $0 \leq \theta < \theta_e$ and imposes $M = 0$ when $\theta_e \leq \theta$. It will also be shown that the optimal output $M^E(\theta)$ is continuous at θ_e. Because the wage w and price p are continuous in θ, and are finite and positive at θ_e, we have $w > pM/N$ when θ is below θ_e but sufficiently close to it. This implies $y^C > 0$ for productivity levels θ where the manufacturing sector is positive but small. This situation will prevail early in the urbanization process.

Proof of Result for A10.2.

From (10.23) we have $\mu(L) = qre^{rL - qM/N}[1 - (1 - e/N)qM/N]$. Differentiation gives

$$\partial\mu/\partial L = qr^2e^{rL - qM/N}\{[1 - (q/N)e^{rL}][1 - (1 - e/N)qM/N] - (1 - e/N)(q/N)e^{rL}\}$$

The last term in braces is always negative. The term $1 - (1 - e/N)qM/N$ is always positive for $0 \leq L < L_{max}$ from the definition of $L_{max} > 0$. A10.2 implies that $1 - (q/N)e^{rL}$ is non-positive at $L = 0$ and negative for $L > 0$. This establishes that $\partial\mu/\partial L < 0$ holds on the interval $0 \leq L < L_{max}$ when A10.2 is satisfied, and thus also on $0 \leq L < \min\{L_{max}, N - e\}$.

Proof of Lemma 10.2.

When evaluated at L = 0, the derivative $\mu(L) + \lambda[L, \beta(\theta)]$ from (10.23) has the same sign as the expression

$$qr(N - e)^{1-\alpha} - [\beta^{1-\alpha} + (1-\alpha)\alpha^{1/(1-\alpha)}\beta^{-\alpha}] \qquad \text{where } \beta \equiv \beta(\theta).$$

Recall that $\theta \geq 0$ implies $\beta \geq \alpha^{1/(1-\alpha)}$. Also, note that $\beta^{1-\alpha} + (1-\alpha)\alpha^{1/(1-\alpha)}\beta^{-\alpha}$ is increasing in β whenever $\beta > \alpha^{(2-\alpha)/(1-\alpha)}$, which is true whenever $\theta \geq 0$.

Define β_e implicitly by $\mu(0) + \lambda(0, \beta_e) \equiv 0$ where $\beta_e \equiv \beta(\theta_e)$. This gives

$$qr(N - e)^{1-\alpha} \equiv \beta_e^{1-\alpha} + (1-\alpha)\alpha^{1/(1-\alpha)}\beta_e^{-\alpha}$$

(a) When part (a) of Lemma 10.2 applies, all $\beta \geq \alpha^{1/(1-\alpha)}$ yield

$$qr(N - e)^{1-\alpha} < \beta^{1-\alpha} + (1-\alpha)\alpha^{1/(1-\alpha)}\beta^{-\alpha}$$

because $\beta^{1-\alpha} + (1-\alpha)\alpha^{1/(1-\alpha)}\beta^{-\alpha}$ is equal to $\alpha(2-\alpha)$ at $\beta = \alpha^{1/(1-\alpha)}$ and is increasing in β. Thus, there is no $\beta_e \geq \alpha^{1/(1-\alpha)}$ with $\mu(0) + \lambda(0, \beta_e) \equiv 0$, and therefore no $\theta_e \geq 0$ with $\mu(0) + \lambda[0, \beta(\theta_e)] = 0$. The fact that $qr(N - e)^{1-\alpha} < \beta^{1-\alpha} + (1-\alpha)\alpha^{1/(1-\alpha)}\beta^{-\alpha}$ holds for all $\beta \geq \alpha^{1/(1-\alpha)}$ implies that the derivative $\mu(L) + \lambda[L, \beta(\theta)]$ in (10.23) is negative at L = 0 for all $\theta \geq 0$. Strict concavity of V^E on the relevant interval implies $L^E(\theta) = 0$ for all $\theta \geq 0$.

(b) When part (b) of Lemma 10.2 applies, there is a unique $\beta_e \geq \alpha^{1/(1-\alpha)}$ with $qr(N - e)^{1-\alpha} = \beta_e^{1-\alpha} + (1-\alpha)\alpha^{1/(1-\alpha)}\beta_e^{-\alpha}$. This occurs because $\beta^{1-\alpha} + (1-\alpha)\alpha^{1/(1-\alpha)}\beta^{-\alpha}$ is equal to $\alpha(2-\alpha)$ at $\beta = \alpha^{1/(1-\alpha)}$, is increasing in β, and goes to infinity as $\beta \to \infty$. Thus, there is a unique $\theta_e \geq 0$ with $\beta_e \equiv \beta(\theta_e)$. When the equality holds in Lemma 10.2(b), we have $\beta_e = \alpha^{1/(1-\alpha)}$ and therefore $\theta_e = 0$. When the inequality holds, we have $\beta_e > \alpha^{1/(1-\alpha)}$ and therefore $\theta_e > 0$. The rest of part (b) follows from strict concavity of V^E on the relevant interval, earlier results about the sign of the derivative from (10.23) at L = 0, the fact that $\beta^{1-\alpha} + (1-\alpha)\alpha^{1/(1-\alpha)}\beta^{-\alpha}$ is increasing in β, and the fact that β is increasing in θ.

Proof of Lemma 10.3.

We use A10.2 here, which implies that A10.1 holds in Proposition 10.2. Assume part (b) of Proposition 10.2 applies, and choose an arbitrary $\theta \in [0, \theta_0)$ so that $L^0(\theta) > 0$. We write the labor input yielding zero profit as $L^0 > 0$ for notational compactness. We will show that whenever A10.2 and A10.3 hold, the derivative in (10.23) has $\mu(L^0) + \lambda(L^0, \beta) < 0$. This rules out boundary solutions of the form $L^E = L^0$ in the elite optimization problem D10.4(a).

Recall from (10.20) that the wage is given by $w(L, \theta) = [\beta(\theta)/(N - e - L)]^{1-\alpha}$. When profit is zero so that $pM = wL$, we have $\beta = (pM/L)^{1/(1-\alpha)}(N - e - L)$. Substituting this into $\lambda(L, \beta)$ from (10.23) gives

$$\lambda = -(N-e-L)^{\alpha-2}[(pM/L)(N - e - L)^{1-\alpha}(N - e - \alpha L) + (1-\alpha)\alpha^{1/(1-\alpha)}(pM/L)^{-\alpha/(1-\alpha)}(N - e - L)^{1-\alpha}]$$

or $$\lambda = -(N - e - L)^{-1}[(pM/L)(N - e - \alpha L) + (1-\alpha)\alpha^{1/(1-\alpha)}(pM/L)^{-\alpha/(1-\alpha)}]$$

We want to show that $\mu + \lambda < 0$ in (10.23), or equivalently $-\lambda > \mu$. The latter holds iff

$$N - e - L + (1-\alpha)L + (1-\alpha)\alpha^{1/(1-\alpha)}(pM/L)^{-1/(1-\alpha)} > (N - e - L)[M'(L)L/M][1 - (1 - e/N)qM/N]$$

Dividing by N - e - L and using (10.14), this is equivalent to

$$1 + (1-\alpha)[L + \alpha^{1/(1-\alpha)}(pM/L)^{-1/(1-\alpha)}]/(N - e - L) > (rLe^{rL})[1 - (1 - e/N)qM/N]/(e^{rL} - 1)$$

Therefore, a sufficient condition for $\mu + \lambda < 0$ in (10.23) is

$$1 \geq (rLe^{rL})[1 - (1 - e/N)qM/N]/(e^{rL} - 1)$$

Abbreviating $rL \equiv K \geq 0$, using $M = e^K - 1$, and reorganizing this condition gives the equivalent statement

$$\sigma(K) \equiv Ke^{2K}(q/N)(1 - e/N) + e^K[1 - K - K(q/N)(1 - e/N)] - 1 \geq 0$$

where $\sigma(0) = 0$. Differentiation yields

$\sigma'(K) =$

$$e^K\{2Ke^K(q/N)(1 - e/N) + e^K(q/N)(1 - e/N) - K[1 + (q/N)(1 - e/N)] - (q/N)(1 - e/N)\}$$

where $\sigma'(0) = 0$. Now write $\sigma'(K) \equiv e^K\eta(K)$ where $\eta(K)$ is in braces above. With some reorganization, this gives

$$\eta(K) = (q/N)(1 - e/N)(e^K - 1) + (q/N)(1 - e/N)K(e^K - 1) + K[e^K(q/N)(1 - e/N) - 1]$$

where $\eta(0) = 0$. Now consider the derivative

$$\eta'(K) = (q/N)(1 - e/N)[(2K + 3)e^K - 1] - 1$$

where $\eta'(0) = 2(q/N)(1 - e/N) - 1$. From A10.2 we have $q/N \geq 1$ and from A10.3 we have $e/N \leq 1/2$. It follows that $\eta'(0) \geq 0$. This implies $\eta'(K) > 0$ for all $K > 0$ because $\eta'(K)$ is strictly increasing in K. Using $\eta(0) = 0$ gives $\eta(K) > 0$ for all $K > 0$, and hence $\sigma'(K) \equiv e^K\eta(K) > 0$ for all $K > 0$. Using $\sigma(0) = 0$ then yields $\sigma(K) > 0$ for all $K > 0$. This implies that $1 > (rLe^{rL})[1 - (1 - e/N)qM/N]/(e^{rL} - 1)$ holds for all $L > 0$. We have shown that the latter is a sufficient condition for $\mu + \lambda < 0$ to hold at $L^0 > 0$. This concludes the proof.

Proof of Lemma 10.4.

We first show $\theta_e < \theta_0$. From (10.18) we have $\beta_0^{1-\alpha} = qr(N - e)^{1-\alpha}$ where $\beta_0 \equiv \beta(\theta_0)$. From Lemma 10.2 we have $\mu(0) + \lambda(0, \beta_e) \equiv 0$, where $\beta_e \equiv \beta(\theta_e)$. This gives $qr(N - e)^{1-\alpha} = \beta_e^{1-\alpha} + (1-\alpha)\alpha^{1/(1-\alpha)}\beta_e^{-\alpha}$. Therefore, $\beta_0^{1-\alpha} = \beta_e^{1-\alpha} + (1-\alpha)\alpha^{1/(1-\alpha)}\beta_e^{-\alpha}$ which implies $\beta_0 > \beta_e$ and therefore $\theta_0 > \theta_e$. From Proposition 10.2(b) this implies that for all $\theta \in [0, \theta_e]$ we must have $L^0(\theta) > 0$ so positive manufacturing labor is feasible in D10.4(a).

As explained in the text, any optimal labor input $L^E(\theta)$ must have $0 \leq L^E(\theta) < \min \{L_{max}, N\text{-}e\}$. Due to A10.2 the elite objective $V^E[p(L), X^E(L, \theta)]$ is strictly concave for labor inputs in this interval. This rules out multiple solutions and guarantees uniqueness of $M^E(\theta)$. We have $M^E(\theta_e) = 0$ because by the construction of θ_e the derivative in (10.23) is zero at $L = 0$, and by strict concavity it is negative for all $L > 0$ in the relevant interval. We have $0 < M^E(\theta)$ for $0 \leq \theta < \theta_e$ because from Lemma 10.2, all such θ imply that the derivative in (10.23) is positive at $L = 0$. We have $M^E(\theta) < M^0(\theta)$ for $0 \leq \theta < \theta_e$ because A10.2, A10.3, and Lemma 10.3 rule out a boundary solution $L^E(\theta) = L^0(\theta)$ in D10.4(a). $M^E(\theta)$ is decreasing on $0 \leq \theta < \theta_e$ due to the negative sign of the derivative in (10.25).

To demonstrate that $M^E(\theta)$ is continuous at θ_e, we show that the optimal solution $L(\theta)$ is continuous at θ_e. A few intermediate steps are required. Let $L(0)$ satisfy $\mu[L(0)] + \lambda[L(0), \beta(0)] \equiv 0$. This labor input sets the derivative of V^E in (10.23) equal to zero when $\theta = 0$. We will show that a unique $L(0)$ exists and has $0 < L(0) < \min \{L_{max}, N\text{-}e\}$. First, note that we only consider $L < N\text{-}e$ because no other levels of L can be feasible in D10.4(a). From $\theta_e > 0$, the derivative in (10.23) is strictly positive at $L = 0$ and $\theta = 0$. This derivative is strictly negative for all $L \geq L_{max}$ and $\theta \geq 0$ due to $\mu(L) \leq 0$ and $\lambda[L,$

$\beta(\theta)] < 0$. If $L_{max} < N\text{-}e$ then continuity of the derivative in (10.23) and the fact that it is decreasing on $[0, L_{max})$ due to A10.2 imply existence of a unique $L(0) \in (0, L_{max})$. If N-e $\leq L_{max}$ then the derivative in (10.23) with $\theta = 0$ approaches negative infinity as $L \to N\text{-}e$, and there is a unique $L(0) \in (0, N\text{-}e)$. In either case we have $0 < L(0) < \min \{L_{max}, N\text{-}e\}$. From the monotonicity of $L(\theta)$ on $0 \leq \theta < \theta_e$, the largest optimal labor input on this interval is $L(0) > 0$.

Now return to the continuity issue. Recall that $L(\theta_e) = 0$. We want to show that for any ε with $0 < \varepsilon < L(0)$, we can find some $\delta > 0$ such that $\theta \in (\theta_e - \delta, \theta_e)$ implies $L(\theta) < \varepsilon$. This will establish the continuity of $L(\theta)$ at θ_e.

From the definition of θ_e we have $\mu(0) + \lambda[0, \beta(\theta_e)] = 0$. This, along with $\varepsilon > 0$ and the strict concavity of the elite's objective on the relevant interval, give $\mu(\varepsilon) + \lambda[\varepsilon, \beta(\theta_e)] < 0$. From the definition of $L(0)$, we have $\mu[L(0)] + \lambda[L(0), \beta(0)] = 0$. Using $\varepsilon < L(0)$ and strict concavity on the relevant internal, this gives $\mu(\varepsilon) + \lambda[\varepsilon, \beta(0)] > 0$. From continuity and monotonicity of λ as a function of β, there is a unique $\beta_d \in (\beta(0), \beta(\theta_e))$ such that $\mu(\varepsilon) + \lambda(\varepsilon, \beta_d) = 0$. From continuity and monotonicity of β as a function of θ, there is a unique $\theta_d \in (0, \theta_e)$ such that $\beta(\theta_d) = \beta_d$. Now set $\delta = \theta_e - \theta_d > 0$. From strict concavity on the relevant interval, the first order condition is sufficient for a solution, and $L = \varepsilon$ satisfies the first order condition at the commons productivity $\theta_d = \theta_e - \delta$. Hence, $L(\theta_d) = \varepsilon$. From the fact that $L(\theta)$ is a decreasing function of θ on the interval $(0, \theta_e)$, we have $L(\theta) < \varepsilon$ for all $\theta \in (\theta_e - \delta, \theta_e)$. This shows that $L(\theta)$ is continuous at θ_e. $M(L)$ is continuous, so $M^E(\theta) \equiv M[L(\theta)]$ is also continuous at θ_e.

Proof of Proposition 10.3 (elite taxation equilibrium).

(a) Suppose $qr(N - e)^{1-\alpha} \leq \alpha$. This implies $\theta_0 \leq 0$ from (10.18). Proposition 10.2(a) applies, and hence for all $\theta \geq 0$ there is a zero-profit equilibrium with $L^0 = 0$. By Proposition 10.2(b), there is no zero-profit equilibrium with $L^0 > 0$ for any $\theta \geq 0$. We have $qr(N - e)^{1-\alpha} < \alpha(2-\alpha)$, so Lemma 10.2(a) applies and there is no $\theta_e \geq 0$. The only feasible labor input in D10.4(a) is zero, so for all $\theta \geq 0$ the outcome is a zero-profit equilibrium with M = 0 and no taxation.

(b) Suppose $\alpha < qr(N - e)^{1-\alpha} \leq \alpha(2-\alpha)$. This implies $0 < \theta_0$ from (10.18). By Proposition 10.2, there is a unique zero-profit equilibrium with $L^0 = 0$ when $\theta_0 \leq \theta$ and a unique zero-profit equilibrium with $L^0 > 0$ when $0 \leq \theta < \theta_0$. If $qr(N - e)^{1-\alpha} < \alpha(2-\alpha)$ then Lemma 10.2(a) applies and there is no $\theta_e \geq 0$. If $qr(N - e)^{1-\alpha} = \alpha(2-\alpha)$ then Lemma 10.2(b) applies and $\theta_e = 0$.

(i) For $0 \leq \theta < \theta_0$, $L^0 > 0$ implies that zero-profit equilibrium would give M > 0. But if Lemma 10.2(a) applies, the derivative in (10.23) is negative at L = 0, and if Lemma 10.2(b) applies, the derivative in (10.23) is zero at L = 0. From A10.2, both cases imply that the derivative is negative for L > 0 due to strict concavity on the relevant interval, as explained in the text. Thus, L = 0 is the unique solution for the elite optimization problem in D10.4(a). The elite can achieve this outcome through sufficiently high taxes, yielding M = 0.

(ii) For $\theta_0 \leq \theta$, $L^0 = 0$ implies that zero profit equilibrium gives M = 0. The only feasible labor input in D10.4(a) is zero. Therefore, for all $\theta_0 \leq \theta$, the outcome is a zero-profit equilibrium with M = 0. The elite can achieve this outcome without taxation.

(c) Suppose $\alpha(2-\alpha) < qr(N - e)^{1-\alpha}$. This implies $0 < \theta_0$ from (10.18). By Proposition 10.2, there is a unique zero-profit equilibrium with $L^0 = 0$ when $\theta_0 \leq \theta$ and a unique zero-profit equilibrium with $L^0 > 0$ when $0 \leq \theta < \theta_0$. Lemma 10.2(b) applies, giving $0 < \theta_e$. Also, Lemma 10.4 gives $0 < \theta_e < \theta_0$. From A10.2, A10.3, and Lemma 10.3, there is never an elite optimum with $L^E = L^0$ in D10.4(a) so we restrict attention to boundary solutions with $L^E = 0$ or interior solutions with $0 < L^E < L^0$.

(i) For $0 \leq \theta < \theta_e$, we have $L^0 > 0$. Lemma 10.2(b) implies that the derivative in (10.23) is positive at L = 0, so the solution has $L^E > 0$. Lemma 10.4 ensures that $0 < M^E(\theta) < M^0(\theta)$. Thus, the elite imposes positive taxes but with M > 0.

(ii) For $\theta_e \leq \theta < \theta_0$, we have $L^0 > 0$. Hence, zero-profit equilibrium would give M > 0. Lemma 10.2(b) implies that the derivative in (10.23) is zero at L = 0 when $\theta = \theta_e$ and negative when $\theta > \theta_e$. In either case, A10.2 implies that the derivative is negative for L > 0 on the relevant interval by strict concavity. Thus, $L^E = 0$ is the unique solution for the problem in D10.4(a). The elite can achieve this outcome through sufficiently high taxes, yielding M = 0.

(iii) For $\theta_0 \leq \theta$, we have $L^0 = 0$. Hence, zero-profit equilibrium would give M = 0. The only feasible labor input in D10.4(a) is zero. Therefore, for all θ_0

$\leq \theta$, the outcome is a zero profit equilibrium with M = 0. The elite can achieve this outcome without taxation.

Proof of Lemma 10.5.

Treating elite utility V^E in (10.26) as a function of $[L(\theta), \theta]$, we differentiate with respect to θ and obtain

$$dV^E[p(L(\theta)), X^E(L(\theta), \theta)]/d\theta = \{\mu[L(\theta)] + \lambda[L(\theta), \beta(\theta)]\}L'(\theta) - (F + L)\partial w/\partial\theta$$

where μ and λ are defined in (10.23). By the envelope theorem, the bracketed coefficient of $L'(\theta)$ is zero at interior solutions for L (see Section 10.4). Thus, the derivative reduces to $-(F + L)\partial w/\partial\theta < 0$, where the inequality follows from $F > 0$ and $L > 0$.

Proof of Lemma 10.6.

Treating commoner utility V^C in (10.27) as a function of [L(θ), θ], we differentiate with respect to θ and obtain

$$dV^C[p(L(\theta)), X^C(L(\theta), \theta)]/d\theta$$

$$= (N-e)\{[-(M/N)\partial p/\partial L + \partial w/\partial L]L'(\theta) + \partial w/\partial \theta\}$$

These computations are straightforward.

Proof of Lemma 10.7.

Treat the social planner's utility V^S in (10.28) as a function of $[L(\theta), \theta]$. Because $V^S = V^E + V^C$, we have $dV^S/d\theta = dV^E/d\theta + dV^C/d\theta$. Summing the derivatives from Lemmas 10.5 and 10.6, and using $N - e = C + F + L$, we obtain

$$dV^S[p(L(\theta)), X^S(L(\theta), \theta)]/d\theta$$

$$= (N-e)[-(M/N)\partial p/\partial L + \partial w/\partial L]L'(\theta) + (\partial w/\partial\theta)C$$

Proof of Proposition 10.4 (local welfare effects).

(a) Consider $dV^E[p(L(\theta)), X^E(L(\theta), \theta)]/d\theta = - (F + L)\partial w/\partial\theta < 0$ from Lemma 10.5.

From (10.20), we have $w(L, \theta) = [\beta(\theta)/(N - e - L)]^{1-\alpha}$. This gives

$$\partial w/\partial\theta = (1-\alpha)\beta(\theta)^{-\alpha}\beta'(\theta)/(N - e - L)^{1-\alpha}$$

From (10.5), $F(w) = (\alpha/w)^{1/(1-\alpha)}$. Substituting for the wage from (10.20), this gives

$$F = \alpha^{1/(1-\alpha)}(N - e - L)/\beta(\theta)$$

Letting $\theta \to \theta_e$ from below and noting that $L(\theta) \to 0$, we have

$$- (F + L)\partial w/\partial\theta \to -(1-\alpha)\alpha^{1/(1-\alpha)}(N - e)^{\alpha}\beta_e^{-\alpha-1}\beta'(\theta_e) < 0$$

(b) Consider

$$dV^C[p(L(\theta)), X^C(L(\theta), \theta)]/d\theta$$

$$= (N-e)\{[-(M/N)\partial p/\partial L + \partial w/\partial L]L'(\theta) + \partial w/\partial\theta\}$$

from Lemma 10.6. As $\theta \to \theta_e$ from below, $L(\theta) \to 0$ and $M^E(\theta) \to 0$, while $\partial p/\partial L$ and $L'(\theta)$ remain finite. Hence, we can confine attention to $(N-e)[(\partial w/\partial L)L'(\theta) + \partial w/\partial\theta]$. The derivatives of the wage are calculated from (10.20). The resulting limits are:

For $\partial w/\partial L$: $(1-\alpha)\beta_e^{1-\alpha}(N - e)^{\alpha-2}$ and

For $\partial w/\partial\theta$: $(1-\alpha)\beta_e^{-\alpha}\beta'(\theta_e)(N - e)^{\alpha-1}$

Next, we compute $L'(\theta)$ and consider its limit as $\theta \to \theta_e$ from below. Using (10.25), computing the required derivatives of μ and λ, and taking limits gives

For $L'(\theta)$: $(1-\alpha)\beta_e^{-\alpha}\beta'(\theta_e)(N - e)^{\alpha-1}[1 - \alpha^{(2-\alpha)/(1-\alpha)}\beta_e^{-1}]/D$

where $D \equiv qr^2[1 - (q/N)(2 - e/N)] - (1-\alpha)(N-e)^{\alpha-2}[2\beta_e^{1-\alpha} + (1-\alpha)\alpha^{1/(1-\alpha)}\beta_e^{-\alpha}]$ < 0 is the limit of the denominator in (10.25). The negativity of D follows from $q/N \geq 1$ in A10.2 along with $e/N < 1$.

Multiplying the limit of $(\partial w/\partial L)L'(\theta) + \partial w/\partial\theta$ by N-e and reorganizing yields

$$dV^C/d\theta \to (1/D)(1-\alpha)(N-e)^{\alpha}\beta_e^{-\alpha}\beta'(\theta_e)\bullet$$

$$\{qr^2[1 - (q/N)(2 - e/N)] - (1-\alpha)(N-e)^{\alpha-2}[\beta_e^{1-\alpha} + \alpha^{1/(1-\alpha)}\beta_e^{-\alpha}]\} > 0$$

(c) Because $V^S = V^E + V^C$, we have $dV^S/d\theta = dV^E/d\theta + dV^C/d\theta$. The limit of $dV^S/d\theta$ as $\theta \to \theta_e$ from below is therefore equal to the limit of the sum $dV^E/d\theta + dV^C/d\theta$. These two limits have already been calculated in parts (a) and (b). Summing the results gives

$$(1/D)(1-\alpha)(N-e)^{\alpha}\beta_e^{-\alpha}\beta'(\theta_e)\bullet$$

$$\{-\alpha^{1/(1-\alpha)}\beta_e^{-1}D + qr^2[1 - (q/N)(2 - e/N)] - (1-\alpha)(N-e)^{\alpha-2}[\beta_e^{1-\alpha} + \alpha^{1/(1-\alpha)}\beta_e^{-\alpha}]\}$$

The factor on the first line is negative due to $D < 0$. Thus, the sign of the limit of $dV^S/d\theta$ as $\theta \to \theta_e$ from below is opposite to the sign of the factor on the second line.

Using the earlier result for D, the definition of β_e in Lemma 10.2, and some algebra shows that the factor on the second line has the same sign as

(S) $$r(N-e)[1 - (q/N)(2 - e/N)][\beta_e + (1-\alpha)\alpha^{1/(1-\alpha)}][\beta_e - \alpha^{1/(1-\alpha)}]$$

$$+ (1-\alpha)[-\beta_e^2 + \alpha^{1/(1-\alpha)}\beta_e + (1-\alpha)\alpha^{2/(1-\alpha)}]$$

Recall that when $\theta_e = 0$ we have $\beta_e = \alpha^{1/(1-\alpha)}$, and when $\theta_e > 0$ we have $\beta_e > \alpha^{1/(1-\alpha)}$. The first line in (S) is zero when $\beta_e = \alpha^{1/(1-\alpha)}$ and negative when $\beta_e > \alpha^{1/(1-\alpha)}$. The second line is positive for $\beta_e = \alpha^{1/(1-\alpha)}$, decreasing in β_e on the relevant range, and negative for large values of β_e. If $\theta_e > 0$ and the second line of (S) is non-positive, (S) is negative overall, and $dV^S/d\theta$ is positive in a neighborhood of θ_e. In this case, total utility moves in the same direction as commons productivity.

The more interesting case arises when $\theta_e > 0$ so $\beta_e > \alpha^{1/(1-\alpha)}$ but the second line of (S) is strictly positive. Fix any such value of β_e. From the definition of θ_e and thus β_e in

Lemma 10.2, we have $qr(N-e)^{1-\alpha} = \beta_e^{1-\alpha} + (1-\alpha)\alpha^{1/(1-\alpha)}\beta_e^{-\alpha}$. In what follows, we maintain the fixed value of β_e and adjust other parameters so that this equation continues to hold. Set q = N so A10.2 holds with equality and e = N/2 so A10.3 holds with equality. Then adjust r to maintain the fixed value of β_e when N changes. These manipulations give

$$r(N-e)[1 - (q/N)(2 - e/N)] = -N^{\alpha-1}(1/2)^{\alpha+1}[\beta_e^{1-\alpha} + (1-\alpha)\alpha^{1/(1-\alpha)}\beta_e^{-\alpha}]$$

Substitute the right hand side of this equation into (S) to obtain

$$(S') \qquad -N^{\alpha-1}(1/2)^{\alpha+1}[\beta_e^{1-\alpha} + (1-\alpha)\alpha^{1/(1-\alpha)}\beta_e^{-\alpha}][\beta_e + (1-\alpha)\alpha^{1/(1-\alpha)}][\beta_e - \alpha^{1/(1-\alpha)}]$$

$$+ (1-\alpha)[-\beta_e^2 + \alpha^{1/(1-\alpha)}\beta_e + (1-\alpha)\alpha^{2/(1-\alpha)}]$$

From the choice of β_e the second line is strictly positive. If A10.2 and A10.3 hold with equality and we adjust r in response to N so that β_e is constant, we obtain the following:

(i) For $N \to 0$, the first line of (S′) approaches $-\infty$ while the second line is constant, so (S′) becomes negative. In this case, $dV^S/d\theta$ is positive near θ_e and aggregate utility moves in the same direction as commons productivity.

(ii) For $N \to \infty$, the first line of (S′) approaches 0 while the second line is a positive constant, so (S′) becomes positive. In this case, $dV^S/d\theta$ is negative near θ_e and aggregate utility moves in the opposite direction to commons productivity.

The assumption of equality in A10.2 and A10.3 is not essential. If (S′) is strictly positive or strictly negative, these restrictions can be relaxed without affecting the argument.

Proof of Lemma 10.8.

(a) If $qr(N - e)^{1-\alpha} < \alpha$ we have $\alpha\gamma(0) > qr(N - e)^{1-\alpha}$ due to $\gamma(0) = 1$. There is no $\theta_s \geq 0$ with $\alpha\gamma(\theta_s) \equiv qr(N - e)^{1-\alpha}$ because $\gamma(\theta)$ is increasing. Now consider the planner's objective in (10.29). Because $\alpha\gamma(\theta) > qr(N - e)^{1-\alpha}$ for all $\theta \geq 0$, the derivative of the objective with respect to L is negative at L = 0 for all $\theta \geq 0$. By strict concavity, $L^S(\theta) = 0$ is uniquely optimal.

(b) If $qr(N - e)^{1-\alpha} = \alpha$ we have $\alpha\gamma(0) = qr(N - e)^{1-\alpha}$ and $\theta_s = 0$ by definition. There is no $\theta_s > 0$ with $\alpha\gamma(\theta_s) \equiv qr(N - e)^{1-\alpha}$ because $\gamma(\theta)$ is increasing. Now consider the planner's objective in (10.29). Because $\alpha\gamma(0) = qr(N - e)^{1-\alpha}$, for $\theta = 0$ the derivative of the objective with respect to L is zero at L = 0, and for all $\theta > 0$ the derivative of the objective with respect to L is negative at L = 0. By strict concavity, in each case $L^S(\theta) = 0$ is uniquely optimal.

(c) If $qr(N - e)^{1-\alpha} > \alpha$ we have $\alpha\gamma(0) < qr(N - e)^{1-\alpha}$. There is a unique $\theta_s > 0$ such that $\alpha\gamma(\theta_s) \equiv qr(N - e)^{1-\alpha}$ because $\gamma(\theta)$ is continuous, increasing, and goes to infinity as $\theta \rightarrow \infty$.

  (i) Consider the planner's objective in (10.29). For $0 \leq \theta < \theta_s$ we have $\alpha\gamma(\theta) < qr(N - e)^{1-\alpha}$. This implies that the derivative of the objective with respect to L is positive at L = 0, which results in an interior solution $L^S(\theta) > 0$. For $\theta_s \leq \theta$ we have $\alpha\gamma(\theta) \geq qr(N - e)^{1-\alpha}$. This implies that the derivative of the objective with respect to L is zero or negative at L = 0, which results in a boundary solution $L^S(\theta) = 0$.

(ii) Treating the first order condition for an interior solution as an identity and using the implicit function theorem gives $dL^S(\theta)/d\theta < 0$ on $0 \leq \theta < \theta_s$ and thus also $dM^S(\theta)/d\theta < 0$ on $0 \leq \theta < \theta_s$. To verify continuity at θ_s, consider the interval $0 \leq \theta < \theta_s$ where $L^S(0) > 0$ is the largest planner optimum on this interval. Recall that $L^S(\theta_s) = 0$. We want to show that for any ε with $0 < \varepsilon < L^S(0)$, we can find some $\delta > 0$ such that $\theta \in (\theta_s - \delta, \theta_s)$ implies $L^S(\theta) < \varepsilon$. This will establish the continuity of $L^S(\theta)$ at θ_s. The derivative of the planner's objective is zero at $L = \varepsilon > 0$ when the commons productivity $\theta = \theta_s - \delta$ satisfies

$$-\alpha\gamma(\theta_s - \delta)(N - e - \varepsilon)^{\alpha-1} + qre^{r\boldsymbol{\varepsilon} - qM(\boldsymbol{\varepsilon})/N} = 0$$

where $\theta_s - \delta < \theta_s$ follows from the definition of θ_s, $\varepsilon > 0$, the fact that the derivative of the planner's objective is decreasing in L for $L \geq 0$ when A10.2 holds, and the fact that $\gamma(\theta)$ is increasing. Strict concavity implies that the first order condition is sufficient, and thus $L^S(\theta_s - \delta) = \varepsilon$. Because $L^S(\theta)$ is decreasing on $0 < \theta < \theta_s$, we have $L^S(\theta) < \varepsilon$ for all $\theta \in (\theta_s - \delta, \theta_s)$. This shows that $L^S(\theta)$ is continuous at θ_s. M(L) is continuous and so $M^S(\theta)$ is also continuous at θ_s.

(iii) If $M^0(\theta) = 0$, the result is trivial from (c)(i) above. Therefore, consider an interior zero-profit equilibrium such that $M^0(\theta) > 0$. The derivative of the planner's objective with respect to L can be written

$$b'(M/N)M' - \alpha\gamma(\theta)(N - e - L)^{\alpha-1}$$

From Section 10.3, an interior zero-profit equilibrium satisfies

$$b'(M/N)(M/L) - \beta(\theta)^{1-\alpha}(N - e - L)^{\alpha-1} = 0$$

Now evaluate the planner's derivative at the values (L^0, M^0) from the latter equation. Increasing returns gives $M' > M/L$. Moreover, $\alpha\gamma(\theta) \leq \beta(\theta)^{1-\alpha}$ holds for all $\theta \geq 0$. Therefore, the planner's derivative must be positive at any interior zero-profit equilibrium, so $L^0(\theta) < L^S(\theta)$ and $M^0(\theta) < M^S(\theta)$.

(iv) If Lemma 10.2(b) applies, then $0 \leq \theta_e$. Lemma 10.4, A10.2, and A10.3 imply $0 \leq \theta_e < \theta_0$. The definition of θ_0 gives $\beta(\theta_0)^{1-\alpha} = qr(N - e)^{1-\alpha}$ and the definition of θ_s gives $\alpha\gamma(\theta_s) = qr(N - e)^{1-\alpha}$, so $\alpha\gamma(\theta_s) = \beta(\theta_0)^{1-\alpha}$. For any $\theta > 0$, $\alpha\gamma(\theta) < \beta(\theta)^{1-\alpha}$. Due to $\theta_0 > 0$, we have $\alpha\gamma(\theta_0) < \beta(\theta_0)^{1-\alpha}$. Because $\alpha\gamma(\theta_0) < \alpha\gamma(\theta_s)$ and $\gamma(\theta)$ is increasing, we have $\theta_0 < \theta_s$.

Proof of Proposition 10.5 (global welfare effects).

The proof of the main result is described in the text. Here we show that one can find parameter values such that (i) the stratification constraints in Section 10.2 hold, (ii) the requirements $0 < \theta_e < \theta_s < \theta_{max}$ are satisfied, and (iii) A10.2 and A10.3 are satisfied.

Recall from Section 10.2 and Figure 10.1 that at $N - e = e[Z + \alpha/(1-\alpha)]$, the stratification constraints are satisfied for all $\theta \in [0, \theta_{max})$ where θ_{max} is defined in (10.10). Furthermore, if these constraints hold for an agricultural equilibrium at a given θ, they will also hold for a manufacturing equilibrium at any lower θ having the same commoner population N-e. Thus, in the context of Proposition 10.5, we only need to show that these constraints are satisfied for the initial agricultural equilibrium where $\theta \in (\theta_s, \theta_{max})$.

Lemma 10.2(b) states that $0 < \theta_e$ iff $\alpha(2-\alpha) < qr(N - e)^{1-\alpha}$. By Lemma 10.8(c)(iv) we automatically have $\theta_e < \theta_s$. From the definition of θ_s in Lemma 10.8 and the fact that $\gamma(\theta)$ is increasing, we have $\theta_s < \theta_{max}$ iff $\alpha\gamma(\theta_{max}) > qr(N - e)^{1-\alpha}$. Putting these together, we have $0 < \theta_e < \theta_s < \theta_{max}$ iff $\alpha(2-\alpha) < qr(N - e)^{1-\alpha} < \alpha\gamma(\theta_{max})$.

Set $N - e = e[Z + \alpha/(1-\alpha)]$ to ensure the stratification constraints are satisfied, as explained above. This takes care of requirement (i). The preceding inequalities reduce to $\alpha(2-\alpha) < qre^{1-\alpha}[Z + \alpha/(1-\alpha)]^{1-\alpha} < \alpha\gamma(\theta_{max})$. We can ensure $\alpha(2-\alpha) < \alpha\gamma(\theta_{max})$ by having Z sufficiently large. Choose any such Z, and then choose $qre^{1-\alpha}$ to give $\alpha(2-\alpha) < qre^{1-\alpha}[Z + \alpha/(1-\alpha)]^{1-\alpha} < \alpha\gamma(\theta_{max})$. This takes care of requirement (ii).

Finally, let A10.2 and A10.3 hold with equality so q = N and e = N/2. In this case we have $qre^{1-\alpha} = rN(N/2)^{1-\alpha}$, so it is possible to choose r and N to satisfy the inequalities at the end of the preceding paragraph. This takes care of requirement (iii).

The simplifications involving equalities for A10.2 and A10.3, along with the choice of the particular commoner population $N - e = e[Z + \alpha/(1-\alpha)]$, are not essential. These can be relaxed to show that Proposition 10.5 applies over a larger region of the parameter space. For brevity, we omit a more general analysis here.

Proof of Lemma 10.9.

Set $L \equiv 0$, ignore D10.5(a), and note that D10.5(b) holds iff $\gamma(\theta)(N - e)^\alpha = N/\rho$ as in Figure 10.5. Consider $\gamma(\theta)(N - e)^\alpha - N/\rho$, which is defined for $N \geq e$, strictly concave, and negative at $N = e$. The ray N/ρ intersects the curve $\gamma(\theta)(N - e)^\alpha$ iff the maximum value of this difference is non-negative. The first order condition for a maximum holds when $N = e + [\alpha\gamma(\theta)\rho]^{1/(1-\alpha)}$. Substituting this into the objective function, the maximum value of the difference is non-negative iff $e \leq (1-\alpha)\alpha^{\alpha/(1-\alpha)}\gamma(\theta)^{1/(1-\alpha)}\rho^{1/(1-\alpha)}$.

If this inequality is violated, N/ρ and $V = \gamma(\theta)(N - e)^\alpha$ do not intersect and there is no LRE. If it holds with equality, there is a unique LRE involving a tangency between N/ρ and $V = \gamma(\theta)(N - e)^\alpha$. This LRE is not stable because V/N attains a maximum at the tangency point and declines when N deviates from the LRE value in either direction, so there is no neighborhood of the LRE on which V/N is decreasing as required in D10.5. An instability arises because with an initial drop in N, population continues to decrease.

When the inequality $e < (1-\alpha)\alpha^{\alpha/(1-\alpha)}\gamma(\theta)^{1/(1-\alpha)}\rho^{1/(1-\alpha)}$ holds strictly, the maximum value of the difference $\gamma(\theta)(N - e)^\alpha - N/\rho$ is positive and there are two intersection points as in Figure 10.5. Both of the associated population levels satisfy the LRE condition in D10.5(b). Population falls when $V < N/\rho$ and rises when $V > N/\rho$ as indicated by the arrows in Figure 10.5. The smaller N level is unstable because V/N is locally increasing. The larger N level is stable because V/N is locally decreasing.

This shows that for some arbitrary $\theta \geq 0$ there is a unique stable LRE if and only if $e < (1-\alpha)\alpha^{\alpha/(1-\alpha)}\gamma(\theta)^{1/(1-\alpha)}\rho^{1/(1-\alpha)}$. Because $\gamma(\theta)$ is increasing, this inequality holds for all $\theta \geq 0$ iff it holds for $\theta = 0$. Using $\gamma(0) = 1$, we obtain the result in Lemma 10.9.

Finally, let $f(\theta) = N - e$ be the commoner population at the unique stable LR. We want to show that this is increasing in θ, implying that it is increasing in γ as claimed in Lemma 10.9. By construction, $[f(\theta) + e]/\rho = \gamma(\theta)f(\theta)^\alpha$ for all $\theta \geq 0$. Differentiation gives

$$f'(\theta) = \gamma'(\theta)f(\theta)^\alpha/\{1/\rho - \alpha\gamma(\theta)f(\theta)^{\alpha-1}\}$$

Because $\gamma'(\theta) > 0$ and $f(\theta) > 0$, the numerator is positive. The denominator is positive if $f(\theta)^{1-\alpha} > \alpha\gamma(\theta)\rho$. The value of N that maximizes the difference $\gamma(\theta)(N - e)^\alpha - N/\rho$ satisfies $(N - e)^{1-\alpha} = \alpha\gamma(\theta)\rho$. However, $f(\theta)$ exceeds this value of $N - e$ (see Figure 10.5). Thus, $f'(\theta) > 0$ holds for all $\theta \geq 0$.

Proof of Lemma 10.10.

(a) For the case $(qr)^0 < \alpha(2-\alpha)/(N^0 - e)^{1-\alpha}$, Lemma 10.2(a) implies $L^E(\theta) = 0$ for all $\theta \geq 0$. For the case $(qr)^0 = \alpha(2-\alpha)/(N^0 - e)^{1-\alpha}$, Lemma 10.2(b) implies $\theta_e = 0$ and again $L^E(\theta) = 0$ for all $\theta \geq 0$.

(b) For the case $(qr)^0 > \alpha(2-\alpha)/(N^0 - e)^{1-\alpha}$, Lemma 10.2(b) implies that there is a unique $\theta_e > 0$ such that (i) $L^E(\theta) > 0$ for $0 \leq \theta < \theta_e$ and (ii) $L^E(\theta) = 0$ for $\theta_e \leq \theta$. We limit attention to cases where θ_e is small enough to satisfy $0 < \theta_e < \theta_{max}$.

<u>Proof of Proposition 10.6</u> (long run transition to manufacturing).

Let $N_{min} \equiv e/(1-\alpha)$ and $N_{max} \equiv q$. Let $e_{max} \equiv (1-\alpha)q$ be the value of e such that $N_{min} = N_{max}$, with $N_{min} < N_{max}$ for $e < e_{max}$. We will only consider values of the parameter e for which the latter inequality holds.

At all stages in the proof, population will remain in the interval $N \in (N_{min}, N_{max}]$. This implies that the elite objective function will be strictly concave in L at any relevant (N, θ), so the elite labor allocation $L^E(N, \theta)$ is unique. For more information on this, see the discussion of A10.2 in the text and the Result for A10.2 earlier in these proofs.

(a) Choose arbitrary values of q and e such that $e/(1-\alpha) < q$. We will impose an upper bound on e in part (c) below and consider 'large' values of q in part (d) below. Let $N^0 \in (e/(1-\alpha), q]$ be the initial total population and let $\gamma^0 \in (1, \gamma_{max})$ be the initial agricultural productivity. As in (10.33), let N^0 be stationary subject to $L \equiv 0$ by use of $\rho\gamma^0 = N^0/(N^0 - e)^\alpha$. For any ρ in the interval $e^{1-\alpha}/\gamma_{max}\theta_{max} < \rho < q/(q - e)^\alpha$ there are choices of $N^0 \in (e/(1-\alpha), q]$ and $\gamma^0 \in (1, \gamma_{max})$ such that $\rho\gamma^0 = N^0/(N^0 - e)^\alpha$. We will fix a value for ρ in part (c) and verify there that $e^{1-\alpha}/\gamma_{max}\theta_{max} < \rho < q/(q - e)^\alpha$ holds. For any such value of ρ we assume that N^0 and γ^0 are chosen to satisfy the requirements of this paragraph.

Now fix these choices of $N^0 \in (e/(1-\alpha), q]$ and $\gamma^0 \in (1, \gamma_{max})$, and note that $\gamma^0 > 1$ implies $\theta^0 > 0$. Choose $r^0 > \alpha(2-\alpha)/q(N^0 - e)^{1-\alpha}$ so that $0 < \theta_e^0$ as in Lemma 10.10(b), while making r^0 small enough that $0 < \theta_e^0 < \theta^0$ or equivalently $1 < \gamma_e^0 < \gamma^0$. This can be done for any $N^0 > e$ because a sufficiently low value of r^0 yields $\theta_e^0 = 0$, a sufficiently high value yields $\theta_e^0 > \theta^0$, and θ_e^0 is continuous and increasing in r^0 over the relevant range. The combination N^0 and $L^E = 0$ is an LRE associated with θ^0 as in D10.5 because N^0 is stationary as in (10.33) and $L^E(N^0, \theta^0) = 0$ is optimal for the elite due to $0 < \theta_e^0 < \theta^0$ as in Lemma 10.10. Moreover, N^0 is stable as in (10.34) because $e/(1-\alpha) < N^0$.

(b) Given (N^0, r^0), let the commons productivity fall to $\theta' = 0$ or equivalently $\gamma' = 1$. Because $0 < \theta_e^0$ or equivalently $1 < \gamma_e^0$, the resulting SRE has $L^E(N^0, \theta') > 0$ as in Section 10.4.

(c) After the climate shock, agricultural productivity is $\gamma' = 1$. Setting $L \equiv 0$, let N' be the stationary population supported by $\rho\gamma' = N'/(N' - e)^\alpha$ as in (10.33), or simply $\rho = N'/(N' - e)^\alpha$ because $\gamma' = 1$. For reasons to be explained in part (d) we need $\rho < 1$. Hence, we need (i) $N'/(N' - e)^\alpha < 1$. We also need (ii) $e/(1-\alpha) < N'$ in order to have stability for N'. There is a non-degenerate interval of N' values on which both (i) and (ii) are satisfied iff $e < (1-\alpha)\alpha^{\alpha/(1-\alpha)}$. Impose this upper bound on e, choose any N' satisfying (i) and (ii), and set $\rho = N'/(N' - e)^\alpha$. It can be shown that these values of ρ and N', along with $\gamma' = 1$ and $e < (1-\alpha)\alpha^{\alpha/(1-\alpha)}$, satisfy the inequality in Lemma 10.9. Recall from part (a) that ρ must satisfy $e^{1-\alpha}/\gamma_{max}\theta_{max} < \rho < q/(q - e)^\alpha$. In part (d) we will choose a value of q large enough that the upper bound exceeds unity, so this constraint can be ignored. For the lower bound we have $\gamma_{max} > 1$ so it suffices to show that $e^{1-\alpha}/\theta_{max} < \rho$. This holds if $e^{1-\alpha}(N' - e)^\alpha/N' < \theta_{max}$. For a fixed N' the left side is increasing in e and equals θ_{max} when $e/(1-\alpha) = N'$. Thus, the desired inequality holds whenever $e/(1-\alpha) < N'$ as in (ii) above.

Finally, we observe that because ρ is the same in both periods and $1 = \gamma' < \gamma^0$, we have $N' < N^0$. We imposed $N^0 \leq q$ in part (a), so we have $N' < q$ here.

For the given value of ρ and the commons productivity $\theta' = 0$, any LRE such that $L^E = 0$ must have the population N'. To rule out an LRE with $L^E = 0$, it suffices to choose r' so $L^E(N', 0) > 0$. Using Lemma 10.10, define the lower bound $r_{min}' \equiv \alpha(2-\alpha)/q(N' - e)^{1-\alpha}$. For $r' \leq r_{min}'$ we have $L^E = 0$ at N', which gives an LRE associated with $\theta' = 0$. For $r' > r_{min}'$ we have $L^E > 0$ at N' and therefore there is no LRE associated with $\theta' = 0$ that has $L^E = 0$. For the desired result in part (c) of Proposition 10.6, it is necessary and sufficient to have $r' > r_{min}'$. We observe that the lower bound r_{min}' depends on q and N'. We fixed N' earlier in this part of the proof but q remains to be chosen in part (d).

(d) Let N' and ρ have the features described in part (c). As discussed there, we will rule out an LRE with $L^E = 0$ by assuming

(i) $$r' > r_{min}' \equiv \alpha(2-\alpha)/q(N' - e)^{1-\alpha}$$

This implies an inequality involving utility levels:

$$U[L^E(N', 0), N', 0] > U(0, N', 0)$$

To see why, consider a social planner who chooses L to maximize $U(L, N', 0)$. We have shown in the text (see Section 10.5) that when $L^E(N', 0) > 0$, as is true when $r' > r_{min}'$, the planner's choice $L^S(N', 0)$ is larger than $L^E(N', 0)$. Moreover, $U(L, N', 0)$ is strictly concave in L when $N' < q$, which was established in part (c), and the derivative of U with respect to L is zero at $L^S(N', 0)$. Thus, $U(L, N', 0)$ is increasing in L on $0 \leq L \leq L^S(N', 0)$, so $U[L^E(N', 0), N', 0] > U(0, N', 0)$ and

$$U[L^E(N', 0), N', 0]/N' > U(0, N', 0)/N' = 1/\rho$$

where the equality follows from the construction of N′ and ρ in part (c).

Next, we choose a value for the parameter q. In particular, we want to have a value for q such that

(ii) $$U[L^E(q, 0), q, 0]/q < 1/\rho$$

The significance of this condition will be discussed below but first we show that it is possible to choose q so that (ii) holds. Using $\gamma' = 1$ and (10.31), the left side of this inequality cannot exceed $q^{\alpha-1} + 1$. In the limit as $q \to \infty$ the latter approaches unity. Thus whenever $1 < 1/\rho$, (ii) must hold at sufficiently large values of q. But $\rho < 1$ holds by construction from part (c). Also recall from part (c) that we need $\rho < q/(q - e)^{\alpha}$. This is true when $q/(q - e)^{\alpha} > 1$, which holds for all sufficiently large values of q. Choose any value of q satisfying the requirements of this paragraph.

Now suppose (i) and (ii) both hold. A previous result gives

$$U[L^E(q, 0), q, 0]/q < 1/\rho < U[L^E(N', 0), N', 0]/N'$$

The continuity of $L^E(N, 0)$ in N and the continuity of U in L and N imply that there is some $N^* \in (N', q)$ such that

$$U[L^E(N^*, 0), N^*, 0]/N^* = 1/\rho$$

The pair (L^*, N^*) with $L^* = L^E(N^*, 0)$ is an LRE associated with $\theta = 0$ as in D10.5. Any such LRE must have $L^E(N^*, 0) > 0$.

Notice that the choice of q was independent of r′. Having chosen N′ as in part (c), and having chosen a value for q that satisfies (ii), we are free to choose r′ to satisfy (i). This rules out an LRE with $L^E = 0$ and completes the proof.